International Journal of Science Academic Research
Vol. 07, Issue 07, pp.1141-1148, July 2026
Available online at http://www.scienceijsar.com
ISSN: 2582-6425

Research Article

A CONCISE PROOF OF FERMAT'S LAST THEOREM AND GENERAL FERMAT (OR BEAL) CONJECTURE

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Received 18th May 2026; **Accepted** 24th June 2026; **Published online** 31st July 2026

Abstract

A concise proof of Fermat's last theorem and general Fermat (or Beal) conjecture are given in this paper. Firstly, the combinatorial approximation method to compute the approximate number of integer solutions to linear Diophantine equation with solution set of integer subsets is presented. Secondly, the approximate number of integer solutions of Pythagoras theorem and Fermat's last theorem are computed to validate the combinatorial approximation method. Finally, the approximate number of integer solutions of general Fermat (or Beal) conjecture (equation) is computed, thus proof of Fermat's last theorem and general Fermat (or Beal) conjecture are provided.

Keywords: General Fermat(or Beal) conjecture, Fermat's last theorem, Combinatorial approximation method, Linear Diophantine equation, Number of solutions, Set of solutions, Subset of integer.

INTRODUCTION

Introduction and Results

The Fermat's Last Theorem was proposed by French mathematician Pierre de Fermat as a theorem(Fermat's conjecture or Fermat's last theorem) around 1637. British mathematician Andrew Wiles proved the Fermat's conjecture in 1994. After studying Fermat's last theorem in 1993, Texas banker and mathematician Andrew Beal proposed his own conjecture (General Fermat conjecture) and named it Beal's conjecture after himself[1].

Consider the equation (see for instance [2])

$$x_1^p + x_2^q = x_3^r \quad (1)$$

where the unknowns (x_1, x_2, x_3, p, q, r) take their values in the set of tuples of positive integers for which x_1,x_2,$x_3$are relatively prime and p,q, r are ≥2.

Define $\chi = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$-1

Beal's conjecture. Assume $\chi < 0$. Either find another solution to equation (1) or prove that there are no further solutions [3].

A related conjecture, due to R. Tijdeman and D. Zagier[3], is:

Tijdeman-Zagier's Conjecture. The equation (1) has no solutions in positive integers (x_1, x_2, x_3, p, q, r) with each p,q and r at least 3 and with x_1,x_2,x_3 relatively prime.

The next conjecture is proposed by H. Darmon and A. Granville [4]:

Fermat-Catalan Conjecture: The set of solutions (x_1, x_2, x_3, p, q, r) with $\chi < 0$ to the equation (1) is finite.

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It is easy to deduce Fermat-Catalan conjecture from the abc conjecture, once one notices that for p,q, r positive integers, the assumption $\chi < 0$ implies $\chi \leq -\frac{1}{42}$.

In 1995, H. Darmon and A. Granville [4] proved unconditionally that for fixed (p,q,r) with$\chi < 0$, and there are only finitely many (x_1, x_2, x_3) satisfying equation (1).

To obtain integer solutions to general Diophantine equations is very difficult, whereas determining the number of integer solutions to such equations is comparatively much easier. Combinatorial methods, combinatorial approximation methods, enumeration methods, and the density transformation method can be employed to determine the number of integer solutions to the Diophantine equations with countable solution sets. By using the combinatorial approximation method, the following results can be established unconditionally.

Theorem 1 Let x_i $(i = 1,2,3)$, p, q and r be positive integers. If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$<1, the approximate number of integer solutions to the equation $x_1^p + x_2^q = x_3^r$ is < 1.

For determining the number of integer solutions of the Diophantine equation, first, we introduce the combinatorial method, approximate combinatorial method, and enumeration method to compute the number of integer solutions of linear Diophantine equations whose solution set is integer set (integer sequence) and is countable. Second, for cases where the solution sets and the number of solutions sets of the variables in linear Diophantine equations are either the same or different, we propose the approximate combinatorial method, enumeration method, and density transformation method to compute an approximate number of integer solutions of the linear Diophantine equations whose solution sets are integer subsets or sequences of integer subsets.

Proof of Lemma

Lemma

Why study linear Diophantine equations? More specifically, why analyze the number of integer solutions of linear

Diophantine equations whose solution sets consist of sequences of integer subsets?

1. The number of integer solutions to linear Diophantine equations can be determined by combinatorial methods.
2. To find integer solutions to nonlinear or higher-order Diophantine equations is very challenging, whereas counting the number of integer solutions for linear Diophantine equations is comparatively easier.
3. The problem of counting the number of integer solutions to nonlinear or higher-order Diophantine equations can be transformed into the problem of counting the number of integer solutions to linear Diophantine equations whose solution sets are integer subsets.
4. The enumeration method allows for straightforward listing of all integer solutions of linear Diophantine equations when the solution set is an integer sequence, or all integer solutions are expressed as numerical equations or equalities.
5. Since all integer solutions of linear Diophantine equations with solution sets formed by sequences of integer subsets belong to the integer solutions of linear Diophantine equations with integer sequences as solution sets, the enumeration method likewise facilitates the straightforward listing of all such solutions as numerical equations or equalities.
6. By utilizing the enumeration method, it is straightforward to list all local integer solutions, or all of equations or equalities of local integer solutions of linear Diophantine equations whose solution sets are sequences of integer subsets. Specifically, linear Diophantine equations can explicitly or concisely be represented by the numerical equivalence relations within each local equation ($x_1 + x_2 + \cdots + x_i + \cdots + x_m = i\ (1 \le i \le N)$).

This section mainly discusses the number of integer solutions to linear Diophantine equations whose countable solution sets are given by integer sets or integer subset (sequences). The definitions and notations used herein are consistent with those in reference [5].

Lemma 1 presents the number of integer solutions of linear Diophantine equations with integer set sequence as the solution set. Lemma 2 provides the approximate number of integer solutions of linear Diophantine equations with integer set sequence as the solution set. Lemma 3 provides the approximate number of integer solutions of linear Diophantine equations when the solution sets and the number of the solution sets of variables in linear Diophantine equations are either the same or different. Lemma 4 and Lemma 5 present the properties of general linear Diophantine equations when transformed into the standard linear Diophantine equations.

Lemma 1 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right)$ be the number of integer solutions of the standard linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

The number of integer solutions is well known as

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) = \binom{N-1}{m-1} = C_{N-1}^{m-1}$$
$$= \frac{1}{(m-1)!}\frac{N(N-1)(N-2)\cdots\left(N-(m-1)\right)}{N}$$

For any non-negative integer N, the solution set is $x_i = \{1,2,3,\cdots,N\}$, and the number of the solution set is $|x_i| = N$.

Lemma 2 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right)$ be the number of integer solutions of the standard linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

then the approximate number of integer solutions to the equation is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) \sim \frac{1}{(m-1)!}\frac{1}{N}N^m$$

For any non-negative integer N, the solution set is $x_i = \{1,2,3,\cdots,N\}$, and the number of the solution set is $|x_i| = N$.

Lemma 3 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right)$ be the number of integer solutions of the standard linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

then the approximate number of integer solutions to the equation is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) \sim \frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m|$$

For any non-negative integer N, the solution set is $x_i \subseteq \{1,2,3,\cdots,N\}$, the solution set can be $x_i \neq x_j$, and the number of the solution set is $|x_i| \le N$, it can be $|x_i| \neq |x_j|$.

Lemma 4 The number of integer solutions of equation $\pm x_1 \pm x_2 \pm \cdots \pm x_i \pm \cdots \pm x_m = N$ is equal to the number of integer solutions of equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$.

Lemma 5 The number of integer solutions of equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = 0$ is equal to the number of integer solutions of equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$.

Proof of the Lemmas

Lemma 1 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right)$ be the number of integer solutions of the standard linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

The number of integer solutions is well known as

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) = \binom{N-1}{m-1} = C_{N-1}^{m-1}$$
$$= \frac{1}{(m-1)!}\frac{N(N-1)(N-2)\cdots\left(N-(m-1)\right)}{N}$$

For any non-negative integer$N$, the solution set$x_i = \{1,2,3,\cdots,N\}$, the number of the solution set $|x_i| = N$.

Proof

Combinatorial Method

Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right)$ be the number of integer solutions of the standard linear Diophantine equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$

It is well known that the number of integer solutions is [6]

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) = \binom{N+m-1}{m-1} = C_{N+m-1}^{m-1} \tag{2}$$

For any non-negative integer N, the solution set is $x_i = \{1,2,3,\cdots,N\}$, and the number of the solution set is $|x_i| = N$.

Substitute $x_i - 1 = y_i$ into the standard linear Diophantine equation to obtain

$$y_1 + y_2 + \cdots + y_i + \cdots + y_m = N - m \tag{3}$$

The number of integer solutions of equation (3) is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) = \binom{N-1}{m-1} = C_{N-1}^{m-1} \tag{4}$$

The number of integer solutions of equation (3) is equal to the number of integer solutions of the standard linear Diophantine equation

$$C_{m+N-1}^{m-1} = C_{N-1}^{m-1} \text{ or } \binom{N+m-1}{m-1} = \binom{N-1}{m-1} \tag{5}$$

According to combinatorial relations, we have

$$C_N^m = \frac{N}{m} C_{N-1}^{m-1} \text{ or } C_{N-1}^{m-1} = \frac{m}{N} C_N^m \tag{6}$$

$$C_N^m - C_{N-1}^m = C_{N-1}^{m-1} \tag{7}$$

Equation (7) presents the recursive relation between the m-dimensional linear Diophantine equation $(x_1 + x_2 + \cdots + x_i + \cdots + x_m = N)$ and the $(m-1)$-dimensional linear Diophantine equation $(x_1 + x_2 + \cdots + x_i + \cdots + x_{m-1} = N)$. This equation (7) also provides the recursive relation between the height N linear Diophantine equation $(x_1 + x_2 + \cdots + x_i + \cdots + x_m = N)$ and the height $(N-1)$linear Diophantine equation $(x_1 + x_2 + \cdots + x_i + \cdots + x_m = N-1)$. Equation (7) can be used, based on height N and dimension m, to prove the number of integer solutions of the linear Diophantine equations by mathematical induction.

Local-Global Principle

The number of integer solutions of the standard linear Diophantine equation is denoted as

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) = \binom{N-1}{m-1} = C_{N-1}^{m-1} = \frac{1}{(m-1)!}\frac{N(N-1)(N-2)\cdots(N-(m-1))}{N} \tag{8}$$

The product $\frac{1}{m!}N(N-1)(N-2)\cdots(N-(m-1))$in the formula $C_N^m - C_{N-1}^m = C_{N-1}^{m-1}$ represents the total number of integer solutions of the equation in m dimensions.

In the product $\frac{1}{m!}\frac{N(N-1)(N-2)\cdots(N-(m-1))}{N}$, the product t$\frac{1}{m!}N(N-1)(N-2)\cdots(N-(m-1))$is divided by N, therefore for each $N(N = 1,2,\cdots,i,\cdots,N)$, if the local equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = i(1 \le i \le N))$ have or not solutions, then the global equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$ also has or not solutions, and vice versa.

Enumeration Method

Linear Diophantine equations can provide clear or concise numerical equivalence relations in each local equation $(x_1 + x_2 + \cdots + x_i + \cdots + x_m = i\ (1 \le i \le N))$.

In linear Diophantine equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$, when $m = 3$, the equation is transformed into $x_1 + x_2 + x_3 = N$. According to lemma 5, the number of integer solutions to the equation $x_1 + x_2 + x_3 = N$is equal to the number of integer solutions to the equation $x_1 + x_2 + x_3 = 0$.

According to Lemma 4, the number of integer solutions to the equation $x_1 + x_2 + x_3 = 0$is equal to the number of integer solutions to the equation $x_1 + x_2 - x_3 = 0$ or $x_1 + x_2 = x_3$.

Now, consider the enumeration method to list the integer solutions of linear Diophantine equations with a countable solution set. From the above, when $m = 3$, the number of integer solutions to the equation $x_1 + x_2 = x_3$ or $x_1 + x_2 + x_3 = N$ is equal to $C_{N-1}^{3-1} = C_{N-1}^2$.

Table 1 below shows the total number of integer solutions of the 3D equation $x_1 + x_2 = x_3$ where $x_i \le 10$.

Table 1. Total number of integer solutions of the 3D equation $x_1 + x_2 = x_3$ where $x_i \le 10$

$x_1 + x_2 = x_3$	1	2	3	4	5	6	7	8	9	10	x_1
1	1+1=2	2+1=3	3+1=4	4+1=5	5+1=6	6+1=7	7+1=8	8+1=9	9+1=10		
2	1+2=3	2+2=4	3+2=5	4+2=6	5+2=7	6+2=8	7+2=9	8+2=10			
3	1+3=4	2+3=5	3+3=6	4+3=7	5+3=8	6+3=9	7+3=10				
4	1+4=5	2+4=6	3+4=7	4+4=8	5+4=9	6+4=10					
5	1+5=6	2+5=7	3+5=8	4+5=9	5+5=10						
6	1+6=7	2+6=8	3+6=9	4+6=10							
7	1+7=8	2+7=9	3+7=10								
8	1+8=9	2+8=10									
9	1+9=10										
10											
x_2											

$x_1 = \{1,2,3,\cdots,10\}$, $x_2 = \{1,2,3,\cdots,10\}$ and $x_3 = \{1,2,3,\cdots,10\}$

The number of integer solutions to the equation $x_1 + x_2 = x_3$ is equal to

$$C_{N-1}^{3-1} = C_{N-1}^{2} = \frac{(N-1)(N-2)}{2!} \quad (9)$$

When $N \le 10$, as shown in Table 1.

When $m = 2$, the number of integer solutions to the equation $x_1 + x_2 = N$ is equal to

$$C_N^2 - C_{N-1}^2 = \frac{(N)(N-1)}{2!} - \frac{(N-1)(N-2)}{2!} = C_{N-1}^1 = N - 1 \quad (10)$$

is indicated by the diagonal in Table 1.

Lemma 1 is proved.

Lemma 2 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right)$ be the number of integer solutions of the standard linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

the approximate number of integer solutions for the equation is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) \sim \frac{1}{(m-1)!}\frac{1}{N}N^m$$

For any non-negative integer N, the solution set is $x_i = \{1,2,3,\cdots,N\}$, and the number of the solution set is $|x_i| = N$.

Proof

Combinatorial approximation method

According to Lemma 1, the equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$has the solution set $x_i = \{1,2,3,\cdots,N\}$, and the number of the solution set is $|x_i| = N$.

We make the following approximation$N(N-1)\cdots\big(N-(m-1)\big) \sim N^m$,$(N-1)(N-2)\cdots(N-m) \sim \langle N-1\rangle^m$ and $(N-1)(N-2)\cdots\big(N-(m-1)\big) \sim N^{\langle m-1\rangle}$.

Obtain

$$\frac{N^m}{m!} - \frac{(N-1)^m}{m!} \sim \frac{N^{(m-1)}}{(m-1)!} \quad (12)$$

Local-Global Principle

The approximate number of solutions to the standard linear Diophantine equation is denoted by

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) \sim \frac{1}{(m-1)!}\frac{N^m}{N} \quad (13)$$

The product $\frac{N^m}{m!}$ in the above formula (12) is the total number of solutions for the m-dimensional equation.

In $\frac{N^m}{m!N}$, the product $\frac{N^m}{m!}$ is divided by N, therefore for each $N(N = 1,2,\cdots,i,\cdots,N)$, if the local equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = i$ $(1 \le i \le N)$ have or not solutions, then the global equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$ also has or not solutions, and vice versa.

Enumeration Method

Equation $x_1 + x_2 = x_3$has the solution set $x_1 = x_2 = x_3 = \{1,2,3,\cdots,N\}$, and the number of solution set is $|x_1| = |x_2| = |x_3| = N$, and the density of the solution set is $d_1 = d_2 = d_3 = 1$.

Table 2 lists the total number of solutions for the 3D equation $x_1 + x_2 = x_3$ satisfies $x_i \le 10$.

Table 2 Total number of solutions of the 3D equation $x_1 + x_2 = x_3$ where $x_i \le 10$

$x_1 + x_2 = x_3$	1	2	3	4	5	6	7	8	9	10	x_1
1	1+1=2	2+1=3	3+1=4	4+1=5	5+1=6	6+1=7	7+1=8	8+1=9	9+1=10		
2	1+2=3	2+2=4	3+2=5	4+2=6	5+2=7	6+2=8	7+2=9	8+2=10	9+2=11		
3	1+3=4	2+3=5	3+3=6	4+3=7	5+3=8	6+3=9	7+3=10	8+3=11	9+3=12		
4	1+4=5	2+4=6	3+4=7	4+4=8	5+4=9	6+4=10	7+4=11	8+4=12	9+4=13		
5	1+5=6	2+5=7	3+5=8	4+5=9	5+5=10	6+5=11	7+5=12	8+5=13	9+5=14		
6	1+6=7	2+6=8	3+6=9	4+6=10	5+6=11	6+6=12	7+6=13	8+6=14	9+6=15		
7	1+7=8	2+7=9	3+7=10	4+7=11	5+7=12	6+7=13	7+7=14	8+7=15	9+7=16		
8	1+8=9	2+8=10	3+8=11	4+8=12	5+8=13	6+8=14	7+8=15	8+8=16	9+8=17		
9	1+9=10	2+9=11	3+9=12	4+9=13	5+9=14	6+9=15	7+9=16	8+9=17	9+9=18		
10											
x_2											

Solution set x_1, x_2 and x_3 is $x_1 = \{1,2,3,\cdots,10\}$, $x_2 = \{1,2,3,\cdots,10\}$ and $x_3 = \{1,2,3,\cdots,10\}$.

The number of integer solutions for this equation is $C_N^m - C_{N-1}^m = C_{N-1}^{m-1}$.

In the formula $C_N^m - C_{N-1}^m = C_{N-1}^{m-1}$, m is finite and relatively N very small; N is very large and infinite.

The formula $C_N^m - C_{N-1}^m = C_{N-1}^{m-1}$ is

$$\frac{N(N-1)\cdots\big(N-(m-1)\big)}{m!} - \frac{(N-1)(N-2)\cdots(N-m)}{m!} = \frac{(N-1)(N-2)\cdots\big(N-(m-1)\big)}{(m-1)!} \quad (11)$$

When $m = 3$, the number of integer solutions of the equation$x_1 + x_2 = x_3$or$x_1 + x_2 + x_3 = N$ is equal to $NS \sim \frac{N^2}{2!}$.

When $m = 2$, the number of integer solutions of the equation $x_1 + x_2 = N$is equal to

$NS \sim \frac{N^2}{2!} - \frac{(N-1)^2}{2!} \sim \frac{1}{1!}N^1$. shown by the diagonal line in Table 2.

The proof of Lemma 2 is completed.

Lemma 3 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right)$ denote the number of integer solutions of the linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

and the approximate number of integer solutions to the equation is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right) \sim \frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m|$$

For any non-negative integer$N$, the solution set is $x_i \subseteq \{1,2,3,\cdots,N\}$, the solution set can be $x_i \neq x_j$, and the number of the solution sets is $|x_i| \leq N$, and it can be $|x_i| \neq |x_j|$.

Proof

Combinatorial approximation method

According to lemma 2, the equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$ has the solution set $x_i = \{1,2,3,\cdots,N\}$, and the number of the solution set is $|x_i| = N$.For the equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$, one has the following approximate number of integer solutions:

$$NS \sim \frac{N^m}{m!} - \frac{(N-1)^m}{m!} \sim \frac{1}{(m-1)!}N^{(m-1)} \tag{14}$$

We rewrite the above expression as

$$\frac{N^m}{m!} - \frac{(N-1)^m}{m!} \sim \frac{1}{(m-1)!}\frac{1}{N}N^m \tag{15}$$

The solution set is $x_i \subseteq \{1,2,3,\cdots,N\}$ and it can be $x_i \neq x_j$, and the number of the solution set is $|x_i| \leq N$, and it can be $|x_i| \neq |x_j|$, and the density of the solution set is $d_i = \frac{|x_i|}{N}$.

The above expression (15) multiplied by the density $(d_1 \cdot d_2 \cdot \cdots \cdot d_i \cdot \cdots \cdot d_m)$ of the solution sets

$$(d_1 \cdot d_2 \cdot \cdots \cdot d_m)\left\{\frac{N^m}{m!} - \frac{(N-1)^m}{m!}\right\} \sim \frac{1}{(m-1)!}\frac{1}{N}(d_1 \cdot d_2 \cdot \cdots \cdot d_m)N^m \tag{16}$$

$$\text{Left side} = \frac{(d_1 \cdot N)(d_2 \cdot N)\cdots(d_m \cdot N)}{m!} - \frac{(d_1 \cdot N - d_1)(d_2 \cdot N - d_2)\cdots(d_m \cdot N - d_m)}{m!} \tag{17}$$

Since $0 < d_i \leq 1$, we make the following approximation $(d_1 \cdot N - d_1)(d_2 \cdot N - d_2)\cdots(d_m \cdot N - d_m) \sim (d_1 \cdot N - 1)(d_2 \cdot N - 1)\cdots(d_m \cdot N - 1)$.

The above formula can be approximated as

Left side$= \frac{(d_1 \cdot N)(d_2 \cdot N)\cdots(d_m \cdot N)}{m!} - \frac{(d_1 \cdot N - 1)(d_2 \cdot N - 1)\cdots(d_m \cdot N - 1)}{m!}$

Left side$= \frac{|x_1||x_2|\cdots|x_m|}{m!} - \frac{(|x_1|-1)(|x_2|-1)\cdots(|x_m|-1)}{m!}$

Right side $= \frac{1}{(m-1)!}\frac{1}{N}(d_1 \cdot d_2 \cdot \cdots \cdot d_m)N^m = \frac{1}{(m-1)!}\frac{1}{N}(d_1 \cdot N)(d_2 \cdot N)\cdots(d_m \cdot N)$

Right side$= \frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m|$

Left side = Right side

$$\frac{|x_1||x_2|\cdots|x_m|}{m!} - \frac{(|x_1|-1)(|x_2|-1)\cdots(|x_m|-1)}{m!} \sim \frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m| \tag{18}$$

Consider the non-uniform distribution of the number of integer solutions caused by the combination of density d_i in each $m$dimension with N change, and then $\frac{1}{(m-1)!}$ is modified to $c(m, d_i)$. $c(m, d_i)$ can be obtained through calculation based on precision; that is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right) \sim \frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m| = c(m, d_i)\frac{1}{N}|x_1||x_2|\cdots|x_m| \tag{19}$$

Local-Global Principle

The approximate number of integer solutions of the standard linear Diophantine equation is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right) \sim \frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m| \tag{20}$$

The product $\frac{|x_1||x_2|\cdots|x_m|}{m!}$in the above formula (18) is the total number of integer solutions in dimension m.

In $\frac{|x_1||x_2|\cdots|x_m|}{Nm!}$, the product $\frac{|x_1||x_2|\cdots|x_m|}{m!}$is divided by N, N ($N = 1,2,\cdots,i,\cdots,N$), if the local equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = i$ $(1 \leq i \leq N)$have or not solutions, then the global equation$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$also has or not solutions, and vice versa.

Enumeration method

Example 1. Prime number

Solution set $p = \{2, 3, 5, \cdots, p_i, \cdots, N\}$, $p_i \leq N$, the number of prime solution sets is $|p| = \frac{N}{\ln N}$, and the density of the prime solution sets is $d_1 = d_2 = \frac{1}{\ln N}$. The approximate number of integer solutions of the equation $p_1 + p_2 = N$is$NS \sim c\frac{|p_1| \cdot |p_2|}{1!N} \sim \frac{0.62N}{(\ln N)^2}$ (including duplicate solutions). See Table 3. For details on Goldbach's conjecture and the number of solutions of Goldbach's equation $p_1 + p_2 = N$, please see reference[5].

Table 3. Total number of integer solutions of 2 Dequation $p_1 + p_2 = N$ where $N \leq 10$

$p_1 + p_2 = N$	2	3	5	7	p_1
2	2+2=4				
3		3+3=6	5+3=8	7+3=10	
5		3+5=8	5+5=10	7+5=12	
7		3+7=10	5+7=12	7+7=14	
p_2					

$x_i = \{2, 3, 5, \cdots, p_i, \cdots, N\} \subset \{1,2,3,\cdots,N\}$,$|x_i| \leq N = 10$

Example 2. Residue class

In the equation $a_1x_1 + a_2x_2 + \cdots + a_ix_i + \cdots + a_mx_m = N$, the solution set of the residue class is $a_ix_i = \{a_i \cdot 1, a_i \cdot 2, \ldots, a_i \cdot k, \ldots, N\}$.

By substitution $a_i x_i = y_i$, we can obtain $y_1 + y_2 + \cdots + y_i + \cdots + y_m = N$.

Solution set y_i is $a_i x_i = \{a_i \cdot 1, a_i \cdot 2, \ldots, a_i \cdot k, \ldots, N\}$, therefore, $a_i \cdot k \le N$, and the number of the solution set is $|a_i x_i| = \frac{N}{a_i}$.

In the equation $x_1 + 2x_2 = x_3$, the solution sets x_1, $2x_2$ and x_3 are

$$x_1 = x_3 = \{1,2,3,\cdots,k,\cdots,N\}$$
$$2x_2 = \{2,4,\cdots,2k,\cdots,N\}$$

The number of the solution set $|x_1| = |x_3| = N, |2x_2| = \frac{N}{2}$.

The number of integer solutions is $NS \sim \frac{|x_1|\cdot|2x_2|\cdot|x_3|}{2!N} = \frac{N\cdot\frac{N}{2}\cdot N}{2N} = \frac{N^2}{4}$. See Table 4.

$\frac{1}{d_i} \cdot y_i = \left\{\frac{1}{d_i}\cdot 1,\ \frac{1}{d_i}\cdot 2,\ \ldots,\ \frac{1}{d_i}\cdot k,\ \ldots,\ N\right\} \subseteq \{1, 2,\ \ 3, \cdots, k, \cdots, N\}$,

and the number of the solution set is $\left|\frac{1}{d_i}\cdot y_i\right| = d_i \cdot N \le N$, the solution set is $\frac{1}{d_i}\cdot y_i$, the density is d_i, and there are

$$\frac{1}{d_1}\cdot y_1 + \frac{1}{d_2}\cdot y_2 + \cdots + \frac{1}{d_i}\cdot y_i + \cdots + \frac{1}{d_m}\cdot y_m = N \qquad (24)$$

According to formula (22), the approximate number of integer solutions of equation (24) is

$$NS = N\left(\frac{1}{d_1}, \frac{1}{d_2}, \cdots, \frac{1}{d_m}; N\right) \sim \frac{m}{N}\frac{1}{(m)!}(d_1\cdot N)(d_2\cdot N)\cdot\ldots\cdot(d_m\cdot N) \qquad (25)$$

where $d_i \cdot N = |x_i|$. The approximate number of integer solutions of equation (24) is also

$$NS \sim \frac{|x_1|\cdot|x_2|\cdots|x_m|}{N\cdot(m-1)!}(N\to\infty) \qquad (26)$$

Table 4. Total number of integer solutions of the equation $x_1 + 2x_2 = x_3$, where $x_{1,\ 3} \le 10, 2x_2 \le 10$

$x_1 + 2x_2 = x_3$	1	2	3	4	5	6	7	8	9	10	x_1
2	1+2=3	2+2=4	3+2=5	4+2=6	5+2=7	6+2=8	7+2=9	8+2=10	9+2=11		
4	1+4=5	2+4=6	3+4=7	4+4=8	5+4=9	6+4=10	7+4=11	8+4=12	9+4=13		
6	1+6=7	2+6=8	3+6=9	4+6=10	5+6=11	6+6=12	7+6=13	8+6=14	9+6=15		
8	1+8=9	2+8=10	3+8=11	4+8=12	5+8=13	6+8=14	7+8=15	8+8=16	9+8=17		
10											
$2x_2$											

$x_i \subseteq \{1,2,3,\cdots,N\},\ \ x_i \ne x_j, |x_i| \ne |x_j|, |x_i| \le N = 10$

Density transformation method

We now discuss the density transformation of linear Diophantine equations whose solution set are integer subsets.

Let $a_1, a_2, \ldots, a_m$ be positive integers to satisfy $gcd(a_1, a_2, \cdots, a_m) = 1$. In addition, let $NS = N(a_1, a_2, \cdots, a_m, N)$ be the number of integer solutions of the equation

$$a_1x_1 + a_2x_2 + \cdots + a_ix_i + \cdots + a_mx_m = N \qquad (21)$$

For non-negative integers $x_1, x_2, \ldots, x_m$, according to Schur's theorem, the approximate number of integer solutions of equation (21) is

$$NS = N(a_1, a_2, \cdots, a_m; N) \sim \frac{N^{m-1}}{(m-1)!\cdot a_1\cdot a_2\cdot\cdots\cdot a_m}(N\to\infty) \qquad (22)$$

For non-negative integers N, provided that N is sufficiently large, especially when there exists an integer N_1 such that each $N \ge N_1$ has at least one representation [6].

Equation $\qquad x_1 + x_2 + \cdots + x_i + \cdots + x_m = N \quad (23)$

Non-negative integers $x_1, x_2, \ldots, x_m$, and the solution sets are integer subsets.
$x_i = \{(a_i)_1, (a_i)_2, (a_i)_3, \cdots, N\} \subseteq \{1,2,3,\cdots,N\} (1 \le i \le m)$, and can be $x_i \ne x_j$.

When $(a_i)_k \le N$, and the density of the solution set is $d_i = \frac{|x_i|}{N}$, and the number of the solution set is $|x_i| = d_i \cdot N \le N$. Transform $x_i = \frac{1}{d_i}\cdot y_i$, the solution set is $y_i = \{1,2,3,\cdots,N\}$ $(1 \le i \le m)$ is an integer set. Solution set

Lemma 3 is proved.

For further discussion on the cases of unequal density $d_i \cdot N = |x_i|$ and applications, see reference [5].

Lemma 4 The number of integer solutions of the equation $\pm x_1 \pm x_2 \pm \cdots \pm x_i \pm \cdots \pm x_m = N$ is equal to the number of integer solutions of the equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$.

Proof

In the equation $\pm x_1 \pm x_2 \pm \cdots \pm x_i \pm \cdots \pm x_m = N \qquad (27)$

By substituting $\pm x_i = y_i$

We can obtain $y_1 + y_2 + \cdots + y_i + \cdots + y_m = N \qquad (28)$

The solution set of negative integers is $-x_i = \{-1, -2, -3, \cdots, -k, \cdots, -N\}$, and when $-N \le -k \le -1$, the number of the solution set is $|-x_i| = N$ or $|-x_i| = |y_i|$.

According to lemma 2 and lemma 3, the approximate number of integer solutions of equation (27) can be expressed as

$$NS \sim \frac{|-x_1|\cdot|-x_2|\cdots|-x_m|}{N\cdot(m-1)!} \sim \frac{N^m}{N\cdot(m-1)!} = \frac{N^{m-1}}{(m-1)!}(N\to\infty) \qquad (29)$$

According to lemma 2 and lemma 3, the approximate number of integer solutions of equation (28) can be expressed as

$$NS \sim \frac{|y_1|\cdot|y_2|\cdots|y_m|}{N\cdot(m-1)!} \sim \frac{N^{m-1}}{(m-1)!}(N\to\infty)(30)$$

The number of integer solutions of equation (27) is equal to the number of integer solutions of equation (28).

Proof of Lemma 4 is completed.

Lemma 5 The number of integer solutions of equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = 0$ is equal to the number of integer solutions of equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$.

Proof

Equation $$x_1 + x_2 + \cdots + x_i + \cdots + x_m = 0 \quad (31)$$
Becomes $$x_1 + x_2 + \cdots + x_i + \cdots + x_m + N = N \quad (32)$$

By substitution $x_i = y_i$ $(1 \le i \le m-1)$, and $x_m + N = y_m$
We can obtain $$y_1 + y_2 + \cdots + y_i + \cdots + y_m = N \quad (33)$$

Since N is a constant; therefore, it is not included the number of the solution set. The number of solution set $|x_m + N|$ is only regarded as $|x_m + N|$=$|x_m|$, $or |x_m + N|$=$|x_m|$=N.

According to lemma 2 and lemma 3, the approximate number of integer solutions of equation (31) can be expressed as

$$NS \sim \frac{|x_1|\cdot|x_2|\cdots|x_m+N|}{N\cdot(m-1)!} = \frac{|x_1|\cdot|x_2|\cdots|x_m|}{N\cdot(m-1)!} \sim \frac{N^m}{N\cdot(m-1)!} = \frac{N^{m-1}}{(m-1)!}(N \to \infty) \quad (34)$$

According to lemma 2 and lemma 3, the approximate number of integer solutions of equation (33) can be expressed as

$$NS \sim \frac{|y_1|\cdot|y_2|\cdots|y_m|}{N\cdot(m-1)!} \sim \frac{N^{m-1}}{(m-1)!}(N \to \infty) \quad (35)$$

The number of integer solutions of equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = 0$ is equal to the number of integer solutions of equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$.

Proof of Lemma 5 is completed.

Proof of Theorem 1

Theorem 1 Let $x_i (i = 1,2,3)$, p,q and r be positive integers. If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$<1, the approximate number of integer solutions to the equation $x_1^p + x_2^q = x_3^r$ is <1.

Proof

Recall lemma 3, when $m = 3$, the equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$change into $x_1 + x_2 + x_3 = N$. The approximate number of integer solutions to the equation $x_1 + x_2 + x_3 = N$ is

$$NS \sim c\frac{|x_1|\cdot|x_2|\cdot|x_3|}{N}(N \to \infty) \quad (36)$$

According to Lemma 4, the number of integer solutions to the equation$x_1 + x_2 = x_3$ is equal to the number of integer solutions to the equation $x_1 + x_2 + x_3 = 0$.

According to Lemma 5, the number of integer solutions to equation $x_1 + x_2 + x_3 = 0$is equal to the number of integer solutions to the equation $x_1 + x_2 + x_3 = N$, while N is a constant of integer.

Transform the equation $x_1 + x_2 = x_3$ to

$$x_1 + x_2 + x_3 = N \quad (37)$$

The approximate number of integer solutions to the equation $x_1 + x_2 + x_3 = N$ or $x_1 + x_2 = x_3$ is

$$NS \sim c\frac{|x_1|\cdot|x_2|\cdot|x_3|}{N} = cN^2(N \to \infty) \quad (38)$$

Where the solution sets are x_i $(i = 1,2,3) = \{1, 2, 3, \cdots, j, \cdots, N\}$, and the number of solution sets is $|x_i|(i = 1,2,3) = N$.

1) Pythagoras theorem and its number of integer solutions

Pythagorean theorem: the equation $x_1^2 + x_2^2 = x_3^2$ has integer solutions.

Because all solution of integer (y_1, y_2, y_3) to the equation $y_1 + y_2 = y_3$contains all solution of integer$(x_1, x_2, x_3)$to the equation$x_1^2 + x_2^2 = x_3^2$, or $(x_1, x_2, x_3) \subset (y_1, y_2, y_3)$, to determine whether or not the equation $x_1^2 + x_2^2 = x_3^2$has solutions, just need to calculate the number of solutions to the equation $x_1^2 + x_2^2 = x_3^2$.

The solution set $x_i^2(i = 1,2,3) = \{1^2, 2^2, 3^2, \cdots, j^2, \cdots, N\}$ is a part of integer set $y_i(i = 1,2,3) = \{1, 2, 3, \cdots, j, \cdots, N\}$.

The number $|x_i^2|(i = 1,2,3)$ of solution sets $x_i^2(i = 1,2,3) = \{1^2, 2^2, 3^2, \cdots, j^2, \cdots, N\}$ is $|x_i^2|(i = 1,2,3)$ =$N^{\frac{1}{2}}$. We have an approximate number of integer solutions to the equation $x_1^2 + x_2^2 = x_3^2$

$$NS \sim \frac{1}{2\,!}\frac{|x_1^2|\times|x_2^2|\times|x_3^2|}{|x_3|} = \frac{1}{2\,!}\frac{N^{\frac{1}{2}}\times N^{\frac{1}{2}}\times N^{\frac{1}{2}}}{N} = \frac{N^{\frac{1}{2}}}{2\,!} > 1 \quad (39)$$

The equation $x_1^2 + x_2^2 = x_3^2$ has integer solutions.

2) Proof of Fermat's conjecture

British mathematician Andrew Wiles proved the Fermat's conjecture in 1994. Here, we apply the above method to study the number of integer solutions to Fermat's theorem.

Fermat's theorem: When an integer $n \ge 3$, there is no solutions of integer to the equation $x_1^n + x_2^n = x_3^n$ regarding integers x_i $(i = 1,2,3)$.

The solution set $x_i^n(i = 1,2,3) = \{1^n, 2^n, 3^n, \cdots, j^n, \cdots, N\}$ is a part of integer set $y_i(i = 1,2,3) = \{1, 2, 3, \cdots, j, \cdots, N\}$.

Because all solutions of integer (y_1, y_2, y_3) to the equation $y_1 + y_2 = y_3$ contains all solution of integer(x_1, x_2, x_3) to the equation $x_1^n + x_2^n = x_3^n$, or $(x_1, x_2, x_3) \subset (y_1, y_2, y_3)$, to determine whether or not the equation $x_1^n + x_2^n = x_3^n$has solutions, just need to calculate the number of solutions to the equation $x_1^n + x_2^n = x_3^n$.

The number $|x_i^n|(i = 1,2,3)$ of solution sets $x_i^n(i = 1,2,3) = \{1^n, 2^n, 3^n, \cdots, j^n, \cdots, N\}$ is $|x_i^n|(i = 1,2,3)$ =$N^{\frac{1}{n}}$.
The number of solutions to the equation $x_1^n + x_2^n = x_3^n$ is equal to the number of solutions to the equation $x_1^n + x_2^n + x_3^n = N$.

We have the approximate number of solutions of the equation $x_1^n + x_2^n + x_3^n = N$

$$NS \sim \frac{1}{2\,!}\frac{|x_1^n|\times|x_2^n|\times|x_3^n|}{|x_3|} = \frac{1}{2\,!}\frac{N^{\frac{1}{n}}\times N^{\frac{1}{n}}\times N^{\frac{1}{n}}}{N} = \frac{N^{\frac{3-n}{n}}}{2\,!}. \quad (40)$$

When $n \geq 3$, the number of integer solutions to the equation $x_1^n + x_2^n = x_3^n$ is less than 1, there are no integer solutions to the equation $x_1^n + x_2^n = x_3^n$.
Proof completed.

3) Proof of Beal's conjecture

After studying Fermat's last theorem in 1993, Texas banker and mathematician Andrew Beal proposed his own conjecture and named it after himself [1].

Beal's conjecture (Fermat-Catalan's conjecture): Let $x_i(i = 1,2,3)$, p, q and r be positive integers. If $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$, the equation $x_1^p + x_2^q = x_3^r$ does not have a solution of integer.

The solution set $x_1^p = \{1^p, 2^p, 3^p, \cdots, j^p, \cdots, N\}$, $x_2^q = \{1^q, 2^q, 3^q, \cdots, j^q, \cdots, N\}$ and $x_3^r = \{1^r, 2^r, 3^r, \cdots, j^r, \cdots, N\}$ is a part of integer set $y_i(i = 1,2,3) = \{1, 2, 3, \cdots, j, \cdots, N\}$.

Because all solutions of integer (y_1, y_2, y_3) to the equation $y_1 + y_2 = y_3$ contains all solutions of integer (x_1, x_2, x_3) to the equation $x_1^p + x_2^q = x_3^r$, or $(x_1, x_2, x_3) \subset (y_1, y_2, y_3)$, to determine whether or not the equation $x_1^p + x_2^q = x_3^r$ has solutions, just need to calculate the number of solutions to the equation $x_1^p + x_2^q = x_3^r$.

The number $|x_1^p|$,$|x_2^q|$ and $|x_3^r|$ of solution sets $x_1^p = \{1^p, 2^p, 3^p, \cdots, j^p, \cdots, N\}$, $x_2^q = \{1^q, 2^q, 3^q, \cdots, j^q, \cdots, N\}$ and $x_3^r = \{1^r, 2^r, 3^r, \cdots, j^r, \cdots, N\}$ is $|x_1^p|=N^{\frac{1}{p}}$,$|x_2^q|=N^{\frac{1}{q}}$ and $|x_3^r|=N^{\frac{1}{r}}$.

The number of solutions to equation $x_1^p + x_2^q = x_3^r$ is equal to the number of solutions to equation $x_1^p + x_2^q - x_3^r = N$ or $x_1^p + x_2^q + x_3^r = N$.

We have the approximate number of solutions to equation $x_1^p + x_2^q + x_3^r = N$

$$NS \sim \frac{1}{2\,!}\frac{|x_1^p|\times|x_2^q|\times|x_3^r|}{|x_3|} = \frac{1}{2\,!}\frac{N^{\frac{1}{p}}\times N^{\frac{1}{q}}\times N^{\frac{1}{r}}}{N} = \frac{N^{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1}}{2\,!} \quad (41)$$

If $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}$-1<0, the number of solutions to the equation $x_1^p + x_2^q = x_3^r$ is

$$NS \sim \frac{1}{2\,!}\frac{|x_1^p|\times|x_2^q|\times|x_3^r|}{|x_3|} = \frac{1}{2\,!}\frac{N^{\frac{1}{p}}\times N^{\frac{1}{q}}\times N^{\frac{1}{r}}}{N} = \frac{N^{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1}}{2\,!} < 1 \quad (42)$$

The equation $x_1^p + x_2^q = x_3^r$ does not have a solution for integers.
Proof completed.

Nevanlinna theory deals with the growth of zeros and poles (roots or solution sets) of meromorphic functions. According to Vojta's conjecture, there is a corresponding (similar) relationship between Nevanlinna theory in function fields and the abc conjecture in number fields (including integers). Since the general Fermat conjecture can be derived from the abc conjecture and concerns the number of zeros (roots or solutions) of equations, Nevanlinna theory, the abc conjecture, Vojta's conjecture, and the general Fermat conjecture all those problem is the same class of problems related to the number of zeros (roots or solutions) of equations. This article presents an approximate combinatorial method for estimating the number of integer solutions to general Diophantine equations with countable solution sets. The applicable countable solution sets include integers, rational numbers, prime numbers, algebraic numbers, polynomials, algebraic number fields, and function fields. Some problems can also be transformed into counting number of solutions of equations. For applications of this method in other areas, such as the essence and number of solutions of the abc conjecture, and so forth, please refer to the reference [5].

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