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Research Article

RIEMANN HYPOTHESIS ERRORS' ESTIMATION & ITS COMMERCIAL A.I. CONTROL SYSTEM SUGGESTION

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Abstract

Error estimation for the E(x) = $\psi(x) - x$ has been a long time discussion hot topic for most of the mathematicians research. In the present paper, this author tries to relate some the significant number theorem(s) and equation(s) between the Riemann Zeta function and the Prime Distributions together with my previous results in the magnified proof in the Riemann Hypothesis – x = 0.5 is the only saddle point among all of the equilibrium points in the critical region 0 < x < 0.9 (in fact these equilibrium points and the only saddle point) are the intersections between $\Re(\zeta(x+yI))$=$\Im(\zeta(x+yI))$,) where $\Re(\zeta(x+yI))$ is continuous and x = 0.5 is unique as well as there is only one pole at x = 1 (this author will prove them in details). At the same time, all of the other points excluding x = 0.5 will oscillate to spread for both sides with the equilibrium point laying in the middle. Certainly, there is also a mirror image converse for the above depicted Riemann Hypothesis picture which will be listed in a table format at the end of this author's conclusion section. Moreover, from the sandwiched logarithmic equations between the Prime Number Theorem's error estimation, we may further employ the Fourier & its Inverse Transform together with the Laplace Transform and its Inverse Transform so as to get the modified Zeta function without the pole "1". By mapping the "1" to e^z through the Z-transform we may further get the wanted "Zeta Filter" and in the mirror inverse Z-transform, we may also find the "Prime Filter". All of these filter may be in practice useful in applying to solve our present cryptography for the encryption and decryption etc. Last but not least, this author will end the present research project with an operator model plus the conclusion paper in cryptography.

Keywords: Artificial Intelligence System, Prime Error Formula Convergence, Riemann Hypothesis Error, Prime Distribution, Novel Fuzzy Fourier Transformation.

INTRODUCTION

The connection issue between the Riemann Zeta function and the Prime Distributions lies in the Riemann Explicit Formula as below:

$$\psi(x)=x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{1}{2} ln(1 - x^{-2}) - ln(2\pi)$$

In fact, we may also express the prime counting function as following:

$$\psi(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{\zeta\prime(s)}{\zeta(s)}\frac{x^s}{s}ds$$ or the so-called Perron's Integral.

By evaluating the above Perron's Integral, we may get:
Pole at s = 1 which will give the main term "x";
Poles at ρ: those non trivial zeta zeros will give the oscillatory terms "$-\sum_{\rho}\frac{x^{\rho}}{\rho}$".

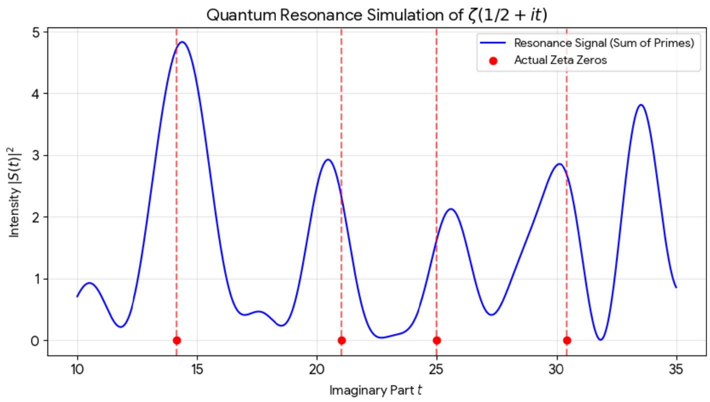
In practice, the error term in the Prime Number Theorem is defined as:

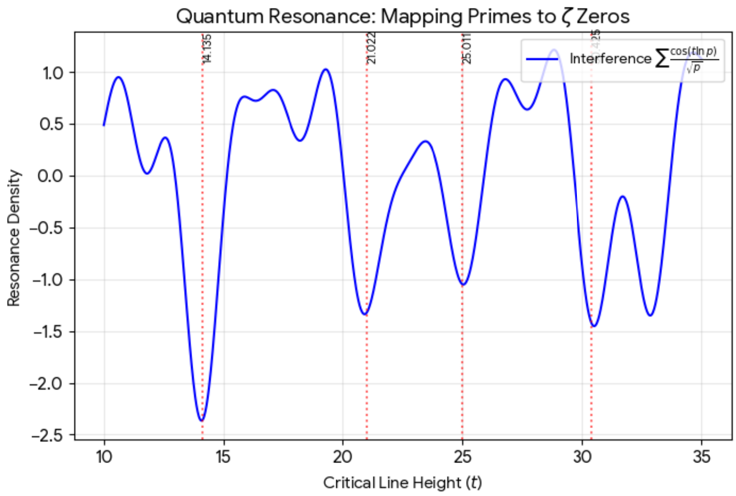
E(x) = $\psi(x) - x$

where its size depends on the real part (β) of the zeros ρ=β+γi.

If we shift the to a vertical line Re(s) = Θ, the error term behaves like O($x^{\Theta+\epsilon}$).

Hence, according to my previous results – Riemann Hypothesis is true as x = 0.5 is just the saddle (or turning) point among all of the equilibrium points in the critical region 0 < x < 0.9 while x = 1 is the only pole. Then the contour will be shifted almost to the critical line and resulted in the smallest possible error estimate: E(x) = O($\sqrt{x}ln(x)$. Moreover, for all of the other equilibrium points excluding the x = 0.5, then the error estimate will be O($x^{\Theta+\epsilon}$) such as O($x^{0.1+\epsilon}$), O($x^{0.7+\epsilon}$) and O($x^{0.9+\epsilon}$) but must NOT be the minimum.





I. A Mathematical (Physics) Preliminary

Consider the following case study:

$$\lim_{x\to 0}\frac{1-cosxcos2xcos3x}{1-cosx}$$

$$= lim_{x\to 0}\frac{1-\left(1-\frac{x^2}{2!}\right)\left(1-\frac{(2x)^2}{2!}\right)\left(1-\frac{(3x)^2}{2!}\right)+o(4)}{1-\left(1-\frac{x^2}{2!}\right)+o(4)}$$

$$= lim_{x\to 0}\frac{1-\left(1-\frac{1}{2}(1+4+9)x^2+\ldots\right)+o(4)}{1-\left(1-\frac{x^2}{2!}\right)+o(4)}$$

$$= lim_{x\to 0}\frac{\frac{1}{2}(1+4+9)x^2}{\frac{x^2}{2}}$$

= 14

In general, we may have:

Suppose f(x) and g(x) are both analytic functions around the open ball neighbour-hood of a point "a" such that f(a) = g(a) = 0, and the two functions also have derivatives at x = a (up to order n-1, where $n \geq 1 \in +\mathbb{Z}$ (i.e. $g^{(n)}(a) \neq 0$). Then the limit of f(x) / g(x) as x→ a is:

According to the Taylor's theorem,

$$f(x) = \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi_f)}{(n+1)!}(x-a)^{n+1}$$

$$g(x) = \frac{g^{(n)}(a)}{n!}(x-a)^n + \frac{g^{(n+1)}(\xi_g)}{(n+1)!}(x-a)^{n+1}$$

where we take the Lagrange form of the remainder, with ξ_f and $\xi_g forsome \mathbb{R} \in (x, a)$.

Thus, we have:

$$lim_{x\to a}\frac{f(x)}{g(x)} = lim_{x\to a}\frac{f^{(n)}(a)+\frac{f^{(n+1)}(\xi_f)}{(n+1)}(x-a)}{g^{(n)}(a)+\frac{g^{(n+1)}(\xi_g)}{(n+1)}(x-a)}$$

$$=\frac{f^{(n)}(a)}{g^{(n)}(a)}.$$

To go ahead a step, let $F'(a) = lim_{x\to a}\frac{f^{(n)}(x)}{g^{(n)}(x)}$, then

$$lim_{x\to a} F'(x)= lim_{x\to a}\frac{F(x)-F(a)}{x-a}$$

$$= lim_{x\to a}\frac{\frac{f^{(n-1)}(x)}{g^{(n-1)}(x)}-\frac{f^{(n-1)}(a)}{g^{(n-1)}(a)}}{x-a}$$

$$= lim_{x\to a}\frac{f^{(n)}(x)}{g^{(n)}(x)}.$$

II. An Analytical Supplementary Proof to the Riemann Hypothesis

Let $f(x) = \sum\frac{x_n}{n^s}$, then consider the following Taylor Expansion of $f(x)$, we have:

$$f(x) = \sum\frac{f^{(n)}(a)}{n!}(x-a)^n+\frac{f^{(n+1)}(\xi_f)}{(n+1)!}(x-a)^{(n+1)}$$

i.e. $\frac{(n!)f(x)}{(x-a)^n} = \sum\frac{k!}{(n-k)!}\frac{f^{(k)}(a)}{(x-a)^k} + \frac{f^{(n+1)}(\xi_f)}{(n+1)!}(x-a)$

Obviously, by taking the limit of the error term $\frac{f^{(n+1)}(\xi_f)}{(n+1)!}(x-a)$ is just zero for "x → a" as:

$$\frac{f^{(n+1)}(\xi_f)}{(n+1)!}(x-a) = \frac{(n!)f(x)}{(x-a)^n}-\sum\frac{k!}{(n-k)!}\frac{f^{(k)}(a)}{(x-a)^k}$$

But $lim_{x\to a}\frac{f^{(n+1)}(\xi_f)}{(n+1)!}(x-a)= 0$, hence we have:

$$lim_{x\to a}.(\frac{(n!)f(x)}{(x-a)^n}-\sum\frac{k!}{(n-k)!}\frac{f^{(k)}(a)}{(x-a)^k}) = 0$$

Take the contour integral for the both side, then we have:

$$\oint\frac{(n!)f(x)}{(x-a)^n}dx= \oint\sum\frac{k!}{(n-k)!}\frac{f^{(k)}(a)}{(x-a)^k}dx$$

$$\oint\sum\frac{k!}{(n-k)!}\frac{f^{(k)}(a)}{(x-a)^k}dx= \sum\frac{k!}{(n-k)!}f^{(k)}(a)\oint\frac{1}{(x-a)^k}dx$$

But the contour line integral of $\oint\frac{1}{(x-a)^k}dx$ is only zero for $k \neq -1$, thus we conclude that: $\oint\sum\frac{k!}{(n-k)!}\frac{f^{(k)}(a)}{(x-a)^k}dx = 0$.

Or by differentiating both sides, we get: $\sum\frac{k!}{(n-k)!}\frac{f^{(k)}(a)}{(x-a)^k} = 0$ converges with a zero error function $\frac{f^{(n+1)}(\xi_f)}{(n+1)!}(x-a)^{(n+1)}$ and $\frac{(n!)f(x)}{(x-a)^n}$ converges to a zero. In other words,

the Taylor Expansion of $f(x) = \sum\frac{x_n}{n^s} = \sum\frac{f^{(n)}(a)}{n!}(x-a)^n+\frac{f^{(n+1)}(\xi_f)}{(n+1)!}(x-a)^{(n+1)}$ converges with an error free tailer term or $f(x) = \sum\frac{x_n}{n^s} = \sum\frac{f^{(n)}(a)}{n!}(x-a)^n$.

(N.B.Alternatively, by taking the contour integral of the error term, we get:

$$\oint\frac{f^{(n+1)}(\xi_f)}{(n+1)!}(x-a)\,dx= \oint\frac{f^{(n+1)}(\xi_f)}{(n+1)!(x-a)^{-1}}dx$$

$= \frac{\left|\left(f^{(n+1)}(\xi_f)\right)\right|}{(n+1)!} \oint \frac{1}{(x-a)^{-1}} dx$

$= (2n\pi I) \frac{\left|\left(f^{(n+1)}(\xi_f)\right)\right|}{(n+1)!}$

which is in practice a constant.

Now, consider the differentiation of the both sides, we get:

$$\partial \oint \frac{f^{(n+1)}(\xi_f)}{(n+1)!}(x-a)\, dx = \partial\left((2n\pi I) \frac{\left|\left(f^{(n+1)}(\xi_f)\right)\right|}{(n+1)!}\right)$$

or $\frac{f^{(n+1)}(\xi_f)}{(n+1)!}(x-a) = 0$).

To sum up, this writer concludes that we can always approximate the Dirichlet Series $f(x) = \sum \frac{x_n}{n^s}$ or the Riemann Zeta function $\sum \frac{1}{n^s}$ by the Taylor Expansion $\sum \frac{f^{(n)}(a)}{n!}(x-a)^n$ even with an error free tailer term.

Then according to the Perron Formula without Error Estimates [],

$$f(x) = \sum \frac{x_n}{n^s} = \sum \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$= \frac{1}{2\pi I} \int_{c-I\infty}^{c+I\infty} f(s+w) \frac{x^w}{w} dw$$

(N.B. The error term for the Taylor Expansion of $f(x) = \sum \frac{x_n}{n^s} = \sum \frac{f^{(n)}(a)}{n!}(x-a)^n$ is just $(2\pi I) \frac{\left|\left(f^{(n+1)}(\xi_f)\right)\right|}{(n+1)!}$, the error term for $\frac{1}{2\pi I} \int_{c-I\infty}^{c+I\infty} f(s+w) \frac{x^w}{w} dw$ is just $\frac{M \log^9 t}{t^2}$ or $(\log|t|)^{10} \frac{x^2}{|t|^2}$).

Hence, the Taylor Expansion for the function $\frac{\zeta'(s)}{\zeta(s)}$ will also have an error free tailer term.

Or in practice,

The error term of $\frac{\zeta'(s)}{\zeta(s)}_{Taylor}$ in general is:

$\frac{\frac{f^{(n+2)}\left(\zeta_{f(n+2)}\right)}{(n+2)!}(x-a)^{(n+2)}}{2\pi I \frac{f^{(n+1)}\left(\zeta_{f(n+1)}\right)}{(n+1)!}(x-a)^{(n+1)}}$. Now, by considering the contour integral of the such error term of $\frac{\zeta'(s)}{\zeta(s)}_{Taylor}$, we will have:

$\oint \frac{M_2(x-a)}{2\pi I M_1(n+2)} ds = \frac{|(M_2)||(x-a)|}{2\pi I |(M_1)||(n+2)|}(0) = 0$. Hence, we may conclude that:

$\Phi_{Taylor}(s) = \int_{a-iT}^{a+iT} 0 * \frac{x^{s+1}}{s(s+1)} ds = 0 * \max(1, x^{a+1}) \frac{(a-1)}{N(N-1)} = 0.$

But the error term of $\frac{\zeta'(s)}{\zeta(s)} \frac{x^{s+1}}{s(s+1)}$, i.e.

$\Phi(s) \ll (\log N)^{10} max(1, x^{a+1}) \frac{(a-1)}{[N(N-1)]}$.

Therefore, this writer has find an error which is equal to a zero (or the best exact value without a small errored value like the $\Phi(s)$) for the $\Phi_{Taylor}(s)$ function when the zeta function is approximated by the Taylor Series Expression instead of the normal calculated one, i.e. $\Phi(s) \ll (\log N)^{10} max(1, x^{a+1}) \frac{(a-1)}{[N(N-1)]}$.

(N.B.1. In general (but NOT for the Discrete Fourier Transform), if we take the contour integral of the Taylor Expression Error term with respect to the (n+k), then we may have:

$\oint \frac{M_2}{2\pi I M_1} \frac{(x-a)}{(n+k)} d(n+k) = \frac{M_2}{M_1}|(x-a)|$.

Now, obviously, we may also find:

$$0 \le \frac{M_2}{M_1}|(x-a)| \le k * \delta = \delta_1$$

which is actually the radius of convergence for the such of the contour integral.

N.B.2. In the sense of engineering, the prime number is linked with the zeta function with the following relationship: $\prod_{p\in prime}\frac{1}{1-p^{-s}}$(by a Discrete Fourier Transform into) $(1-2^{-s})\zeta(s)=\eta(s)$.

In general, the Riemann Zeta function can be viewed as the subset of the Hurwith Zeta function, which is defined as:

$$\zeta(s,q)=\sum_{n=0}^{\infty}\frac{1}{(q+n)^s}.$$

Without lost of generality, for a given sequence of N complex numbers, say, $\{x_0, x_1, x_2, \ldots, x_{N-1}\}$, this sequence of the complex number can be transformed by the discrete Fourier Transform into an N-periodic sequence of complex numbers with:

$$x_k=\sum_{n=0}^{N-1}x_n * e^{\frac{-I2\pi kn}{N}}$$

But in fact, the discrete Fourier Transform of the Hurwitz Zeta function with respect to the order of s is the Legendre chi function, which is defined as:

$$X_s(z)=\sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)^s}$$

In the case of the Zeta function, this is just the $\sum_{k=0}^{\infty}\frac{1}{(2k+1)^s}$ (for a practically computer simulation case study) which can be expressed in terms of Taylor Series by the Canadian Maple Soft (up to the order of 5 for a case study):

$$\sum_{k=0}^{N-1}\frac{1}{e^{((u+v*I)*ln(2*k+1))}}-\frac{2*(u+v*I)*(n-k)}{\left((2*k+1)*e^{((u+v*I)*ln(2*k+1))}\right)}+\frac{\left(\frac{-2*(-v^2+2*I*u*v-v*I+u^2-u)}{(2*k+1)^2}+\frac{4*(u+v*I)^2}{(2*k+1)^2}\right)*(n-k)^2}{e^{((u+v*I)*ln(2*k+1))}}+$$
$$\frac{\left(\frac{-4*(-3*u*v^2+3*v^2+3*I*u^2*v-v^3*I-6*I*u*v+u^3+(2*v)*I-3*u^2+2*u)}{(3*(2*k+1)^3)}+\frac{4*(-v^2+2*I*u*v-v*I+u^2-u)*(u+v*I)}{(2*k+1)^3}-\frac{4*(-v^2+2*I*u*v+v*I+u^2+u)*(u+v*I)}{(2*k+1)^3}\right)*(n-k)^3}{e^{((u+v*I)*ln(2*k+1))}}+$$
$$\frac{\left(\frac{-2*(-6*u^2*v^2+v^4+18*u*v^2+4*I*u^3*v+6*I*v^3-11*v^2-4*I*u*v^3-18*I*u^2*v+u^4+22*I*u*v-6*u^3-(6*v)*I+11*u^2-6*u)}{(3*(2*k+1)^4)}+\frac{8*(-3*u*v^2+3*v^2+3*I*u^2*v-v^3*I-6*I*u*v+u^3+(2*v)*I-3*u^2+2*u)*(u+v*I)}{(3*(2*k+1)^4)}-\frac{4*(-v^2+2*I*u*v+v*I+u^2+u)*(-v^2+2*I*u*v-v*I+u^2-u)}{(2*k+1)^4}+\frac{8*(-3*v^2+3*I*u^2*v-v^3*I+6*I*u*v+u^3-3*u*v^2+(2*v)*I+3*u^2+2*u)*(u+v*I)}{(3*(2*k+1)^4)}\right)*(n-k)^4}{e^{((u+v*I)*ln(2*k+1))}}$$
$$=$$
$$\sum_{k=0}^{N-1}e^{-((u+v*I)*ln(2*k+1))}\left(1-\frac{2*(u+v*I)*(n-k)}{((2*k+1))}+\left(\frac{-2*(-v^2+2*I*u*v-v*I+u^2-u)}{(2*k+1)^2}+\frac{4*(u+v*I)^2}{(2*k+1)^2}\right)*\right.$$
$$(n-k)^2+\left(\frac{-4*(-3*u*v^2+3*v^2+3*I*u^2*v-v^3*I-6*I*u*v+u^3+(2*v)*I-3*u^2+2*u)}{(3*(2*k+1)^3)}+\frac{4*(-v^2+2*I*u*v-v*I+u^2-u)*(u+v*I)}{(2*k+1)^3}-\frac{4*(-v^2+2*I*u*v+v*I+u^2+u)*(u+v*I)}{(2*k+1)^3}\right)*$$
$$(n-k)^3+\left(\frac{-2*(-6*u^2*v^2+v^4+18*u*v^2+4*I*u^3*v+6*I*v^3-11*v^2-4*I*u*v^3-18*I*u^2*v+u^4+22*I*u*v-6*u^3-(6*v)*I+11*u^2-6*u)}{(3*(2*k+1)^4)}+\frac{8*(-3*u*v^2+3*v^2+3*I*u^2*v-v^3*I-6*I*u*v+u^3+(2*v)*I-3*u^2+2*u)*(u+v*I)}{(3*(2*k+1)^4)}-\right.$$
$$\left.\left.\frac{4*(-v^2+2*I*u*v+v*I+u^2+u)*(-v^2+2*I*u*v-v*I+u^2-u)}{(2*k+1)^4}+\frac{8*(-3*v^2+3*I*u^2*v-v^3*I+6*I*u*v+u^3-3*u*v^2+(2*v)*I+3*u^2+2*u)*(u+v*I)}{(3*(2*k+1)^4)}\right)*(n-k)^4\right)$$

and is obviously in the Discrete Fourier Transform format.)

Or to be precise,
$\prod_{p\in prime}\frac{1}{1-p^{-s}}$(by Discrete Fourier Transform into) ζ(s)
ζ(s) =

$$\frac{1}{(1-2^{-s})}*\sum_{k=0}^{N-1}e^{-((u+v*I)*ln(2*k+1))}\left(1-\frac{2*(u+v*I)*(n-k)}{((2*k+1))}+\left(\frac{-2*(-v^2+2*I*u*v-v*I+u^2-u)}{(2*k+1)^2}+\frac{4*(u+v*I)^2}{(2*k+1)^2}\right)*\right.$$
$$(n-k)^2+\left(\frac{-4*(-3*u*v^2+3*v^2+3*I*u^2*v-v^3*I-6*I*u*v+u^3+(2*v)*I-3*u^2+2*u)}{(3*(2*k+1)^3)}+\frac{4*(-v^2+2*I*u*v-v*I+u^2-u)*(u+v*I)}{(2*k+1)^3}-\frac{4*(-v^2+2*I*u*v+v*I+u^2+u)*(u+v*I)}{(2*k+1)^3}\right)*$$
$$(n-k)^3+\left(\frac{-2*(-6*u^2*v^2+v^4+18*u*v^2+4*I*u^3*v+6*I*v^3-11*v^2-4*I*u*v^3-18*I*u^2*v+u^4+22*I*u*v-6*u^3-(6*v)*I+11*u^2-6*u)}{(3*(2*k+1)^4)}+\frac{8*(-3*u*v^2+3*v^2+3*I*u^2*v-v^3*I-6*I*u*v+u^3+(2*v)*I-3*u^2+2*u)*(u+v*I)}{(3*(2*k+1)^4)}-\right.$$
$$\left.\left.\frac{4*(-v^2+2*I*u*v+v*I+u^2+u)*(-v^2+2*I*u*v-v*I+u^2-u)}{(2*k+1)^4}+\frac{8*(-3*v^2+3*I*u^2*v-v^3*I+6*I*u*v+u^3-3*u*v^2+(2*v)*I+3*u^2+2*u)*(u+v*I)}{(3*(2*k+1)^4)}\right)*(n-k)^4\right)$$

which is what we may expect for the present (engineering) reality in the Discrete Fourier Transform and thus we may establish a case study connection between the prime number and the Riemann Zeta function.)
In the mirror image (or the inverse Discrete Fourier Transform) of the above Taylor Series expression of ζ(s), we may get:

$$\prod_{p\in prime}\frac{1}{1-p^{-s}}=$$
$$\frac{1}{(1-2^{-s})}*\sum_{k=0}^{N-1}e^{((u+v*I)*ln(2*k+1))}\frac{2*(u+v*I)*(Dirac(u+v*I)*k+Dirac(1,u+v*I)*I)}{(2*k+1)}+\frac{2*(-v^2+2*I*u*v+v*I+u^2+u)*\left(2*I*Dirac(1,u+v*I)*k+k^2*Dirac(u+v*I)-Dirac(2,u+v*I)\right)}{(2*k+1)^2}+\frac{1}{3(2k+1)^3}4*$$
$$(-3*v^2+3*I*u^2*v-v^3*I+6*I*u*v+u^3-3*u*v^2+(2*v)*I+3*u^2+2*u)*(3*I*Dirac(1,u+v*I)*k^2+Dirac(u+v*I)*k^3-$$
$$Dirac(3,u+v*I)*I-3*Dirac(2,u+v*I)*k)+\frac{1}{3(2k+1)^4}2*(4*I*u^3*v-4*I*u*v^3+18*I*u^2*v-6*I*v^3+u^4-6*u^2*v^2+v^4+22*I*u*$$
$$v+6*u^3-18*u*v^2+(6*v)*I+11*u^2-11*v^2+6*u)*\left(4*I*Dirac(1,u+v*I)*k^3+Dirac(u+v*I)*k^4-4*I*Dirac(3,u+v*I)*k-6*\right.$$
$$\left.Dirac(2,u+v*I)*k^2+Dirac(4,u+v*I)\right)$$

which is my proposed model equation for the Riemann Zeta function expressed by the product of prime numbers.By the way, after simplification, we may get:

$$\frac{1}{3*(2*k+1)^4}\Bigg((-276+192*k^2+(48-528*I)*k)*Dirac(1,u+v*I)+(264-6*k^2+(-3+144*I)*k)*Dirac(2,u+v*I)+((-2*I*k-48)-I)$$
$$*Dirac(3,u+v*I)-192*Dirac(u+v*I)*\left(k^3*I+\left(\frac{13}{8}+\frac{3*I}{4}\right)*k^2+\left(\frac{1}{4}-\frac{5*I}{4}\right)*k-\frac{3}{16}+\frac{I}{32}\right)\Bigg)$$

If we solve for the values of k for each section of the Dirac functions, we may obtain:

1. solve(-276 + 192*k^2 + (48 - 528*y)*k, k)

$k = \frac{-1}{8} + \frac{11*y}{8} + \frac{\sqrt{(-28-22*y)}}{8}$ or $k = \frac{-1}{8} + \frac{11*y}{8} - \frac{\sqrt{(-28-22*y)}}{8}$

2. solve(264 - 6*k^2 + (-3 + 144*y)*k, k)
$k = \frac{-1}{4} + 12 * y - \frac{\sqrt{(-1599-96*y)}}{4}$ or $k = \frac{-1}{4} + 12 * y + \frac{\sqrt{(-1599-96*y)}}{4}$

3. solve(k^3*y + (13/8 + 3*y/4)*k^2 + (1/4 - 5*y/4)*k - 3/16 + y/32, k)
$$k = \frac{\left(-486+2159*y+12*\sqrt{(-13236+2439*y)}\right)^{\left(\frac{1}{3}\right)}}{24} + \frac{\left(\frac{107}{24}-\frac{9*y}{2}\right)}{\left(-486+2159*y+12*\sqrt{(-13236+2439*y)}\right)^{\left(\frac{1}{3}\right)}} - \frac{1}{4} + \frac{13*y}{24}$$
or
$$k = \frac{-\left(-486+2159*I+12*\sqrt{(-13236+2439*I)}\right)^{\left(\frac{1}{3}\right)}}{48} + \frac{\left(\frac{-107}{48}+\frac{9*I}{4}\right)}{\left(-486+2159*I+12*\sqrt{(-13236+2439*I)}\right)^{\left(\frac{1}{3}\right)}} - \frac{1}{4} + \frac{(13*I;)}{24} + \frac{\sqrt{(3)}*\left(\frac{\left(-486+2159*I+12*\sqrt{(-13236+2439*I)}\right)^{\left(\frac{1}{3}\right)}}{12} + \frac{\left(\frac{-107}{12}+9*I\right)}{\left(-486+2159*I+12*\sqrt{(-13236+2439*I)}\right)^{\left(\frac{1}{3}\right)}}\right)*I}{4}$$
$$k = \frac{-\left(-486+2159*I+12*\sqrt{(-13236+2439*I)}\right)^{\left(\frac{1}{3}\right)}}{48} + \frac{\left(\frac{-107}{48}+\frac{9*I}{4}\right)}{\left(-486+2159*I+12*\sqrt{(-13236+2439*I)}\right)^{\left(\frac{1}{3}\right)}} - \frac{1}{4} + \frac{(13*I;)}{24} - \frac{\sqrt{(3)}*\left(\frac{\left(-486+2159*I+12*\sqrt{(-13236+2439*I)}\right)^{\left(\frac{1}{3}\right)}}{12} + \frac{\left(\frac{-107}{12}+9*I\right)}{\left(-486+2159*I+12*\sqrt{(-13236+2439*I)}\right)^{\left(\frac{1}{3}\right)}}\right)*I}{4}$$

4. solve((-2*I*k - 48) - I, k)
$k = \frac{-1}{2} + 24 * I$

In the next section, this writer will use the above tools and provide an analytic proof to the Prime Number Theorem (PNT).

III. An Analytic Proof to the Prime Number Theorem (PNT)

$\Phi(s) = \frac{\zeta'(s)}{\zeta(s)} \frac{x^{s+1}}{s(s+1)}$

Lemma 1: Let a_1, a_2, … be any real sequence and let $s(x) = \sum_{n \leqslant x} a_n$. Further, let $f(x)$ be a real function with continuous derivative $f'(x)$ $\forall \mathbb{R}>0$. Then

$\sum_{n \leqslant x} a_n f(n) = s(x) f(x) - \int_1^x s(u) f'(u) du.$

Lemma 2: For $\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$, $log\zeta(s) = -\sum_p log\left(1 - \frac{1}{p^s}\right) = \sum_p \sum_{j=1}^{\infty} \frac{1}{j} p^{-js}$.
(N.B. By letting $z = \frac{1}{p^s}$ and taking the contour integral over z, we may get:
$\prod_p \oint (1-z)^{-1} dz = \prod_p 2\pi I = 2\pi I\left(lim_{p\to\infty} \prod p\right). \leq 2\pi I p^n$ where p, n → ∞).

Theorem 3: For any s = σ + I t with $\sigma \geq 1$ and $|t| \gg 1$, we have:

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \ll (log|t|)^{10}$$

Theorem 4: For any x > 0, we have:

$$\int_0^x \psi(u) du = -\frac{1}{2\pi I} \oint_C \frac{\zeta'(s)}{\zeta(s)} \frac{x^{s+1}}{s(s+1)} ds$$

where C demotes the straight line a + I t with a > 1 and -∞< t <∞

Theorem 5: We have $\int_0^x \psi(u)\, du \sim \frac{1}{2}x^2$ as x → ∞.

(N.B. We may directly substitute $\xi(s) = \sum \frac{f^{(n)}(a)}{n!}(x-a)^n$ or $\xi(s) = \frac{1}{2\pi I}\int_{c-I\infty}^{c+I\infty} f(s+w)\frac{x^w}{w} dw$ into the equations of the above theorems respectively for computing or proving the corresponding Prime Number Theorem. That say:

$\frac{\zeta\prime(s)}{\zeta(s)} = \frac{\left(\sum\frac{f^{(n)}(a)}{n!}(x-a)^n\right)\prime}{\sum\frac{f^{(n)}(a)}{n!}(x-a)^n}$ or $\frac{\zeta(s)\prime}{\zeta(s)} = \frac{1}{2\pi I}\int_{c-I\infty}^{c+I\infty} f(s+w)\frac{x^w}{w}dw$ where $f(x) = \sum\frac{1}{n^s}$.)

Theorem 6: (Prime Number Theorem) We have:

$\pi(x) \sim \frac{x}{logx}$ as $x \to \infty$

(N.B. With the similar argument and by the Riemann-Lebesgue Lemma in the theory of the Fourier series,

$lim_{x\to\infty}\int_{-\infty}^{+\infty} f(t)e^{Itx}\,dt=0$ if the integral $\int_{-\infty}^{+\infty}|f(t)|\,dt$ converges. This fact implies the integral

$\int_{-\infty}^{+\infty}|h(c+It)|\,dt$ also converges if c > 1, so the integral

$$lim_{x\to\infty}\frac{x^{c-1}}{2\pi}\int_{-\infty}^{+\infty} h\,(c+It)e^{Itlogx}\,dt=0 \qquad \text{where } h(s) = \frac{1}{s(s+1)}\left(-\frac{\zeta(s)\prime}{\zeta(s)} - \frac{1}{s-1}\right)$$

$\psi_1(x) \sim \frac{x^2}{2}$ where $\psi(x)=\sum_{n\leqslant x}\wedge(n)$ and $\psi_1(x)=\int_1^x\psi(t)\,dt$.

Certainly, in practice, we may convert the integral $\int_{-\infty}^{\infty} h\,(c+It)e^{Itlogx}dt$ into a Fourier series like the case as $\sum_{n=1}^{\infty}\big(h(c+It)+h(-c-It)\big)e^{Itlogx}$ or just $\sum_{n=1}^{\infty}|h(c+It)|e^{Itlogx}$.

A Simple Alternative Proof to the Prime Number Theorem:

$\frac{1}{1+ln(j+1)} < \frac{1}{\left(\frac{1+ln(j+1)}{j+1}\right)} < p_j < \frac{1}{ln(j)} < \frac{j}{ln(j)}$

i.e. $\frac{j+1}{1+ln(j+1)} < p_j < \frac{j}{ln(j)}$.

But $\frac{j+1}{1+ln(j+1)} = \frac{j\left(1+\frac{1}{j}\right)}{1+lnj+ln\left(1+\frac{1}{j}\right)} = \frac{j}{1+ln(j)}$

Or $\frac{j}{1+ln(j)} < p_j < \frac{j}{ln(j)}$,

i.e. $\frac{j}{1+ln(j)} = \frac{j}{j\left(\frac{1}{j}+\frac{lnj}{j}\right)} = \frac{j}{ln(j)} < p_j < \frac{j}{ln(j)}$

By taking the limit to the both sides of the above inequality and replace j with n, we may get:

$p_n \sim \frac{n}{lnn}$ which is in fact an equivalent to the first part of the Prime Number Theorem: $\pi(x) \sim \frac{x}{lnx}$.

On the other way, $\frac{1}{1+ln(j+1)} = \frac{1}{(j+1)\left(\frac{1}{j+1}+\frac{ln(j+1)}{(j+1)}\right)} = \frac{1}{ln(j+1)}$

$> -\frac{j+1}{ln(j+1)}. \approx. -\frac{j}{ln(j)}$

i.e. $-\frac{j}{ln(j)} < P_j < \frac{j}{ln(j)}$

Or the radius of the convergence of P_j is just $\frac{j}{ln(j)}$.

Similarly, we have $-\frac{j+1}{ln(j+1)} < P_{j+1} < \frac{j+1}{ln(j+1)}$ and… $-\frac{j+n}{ln(j+n)} < P_{j+n} < \frac{j+n}{ln(j+n)}$

But $\frac{n\left(\frac{j}{n}+1\right)}{lnn\left(\frac{j}{n}+1\right)} = \frac{n}{ln(n)}$ as $n \to \infty$ for $\frac{j}{n} \to 0$ and also $-\frac{n\left(\frac{j}{n}+1\right)}{lnn\left(\frac{j}{n}+1\right)} = -\frac{n}{ln(n)}$,

hence the radius of convergence of (the sequence Prime Number) P_n in the Prime Number Theorem is also $\frac{n}{ln(n)}$.
Then the next part is the computation of the error from the analytic properties of my prime number inequalities.

IV. A Commercial Way of Fuzzy Fourier Series that induces an Engineering Control System

Suppose there is the Universe absolute time for the two space objects A & B, that are generally moving in the Einstein General Relative sense. Then their absolute relative time with respect to each other is: $\frac{\frac{t_{RelObjA}}{t_{\text{abs}}}}{\frac{t_{RelObjB}}{t_{\text{abs}}}}$. At the same time, suppose it is also true for the absolute universe time in the quantum mechanics sense, says, t_{abs}. Hence, by equating both of the qualities, we get:

$$\frac{\frac{t_{RelObjA}}{t_{abs}}}{\frac{t_{RelObjB}}{t_{abs}}} = t_{abs}$$

i.e. $\frac{t_{RelObjA}}{t_{RelObjB}}$ ------- (1)

$$\frac{t_{RelObjA}}{t_{abs}} \frac{t_{abs}}{t_{RelObjB}} = t_{abs}$$

i.e. $\frac{t_{RelObjA} t_{abs}}{t_{RelObjB}} = t_{abs}^{2}$ -------- (2)

Now suppose the fuzzy empirical equation fro the switching between General Relativity time and the Quantum Mechanics time (inspired from the Buffalo University web page for fuzzy logic) to be:

$\Delta t_{switch} = 1 + k \Delta t$ (linear form) -------- (3)

$\Delta t_{switch} = 1 + \gamma^2 (\Delta t)^2$ (quadratic form) ----- (4)

By substituting (3) into (4), we have:

$\Delta t_{switch} = 1 + \gamma^2 (1 + k \Delta t)^2$ --------- (5)

Now, take the $\Delta t = \frac{t_{RelObjA}}{t_{RelObjB}}$ and substitute back into (5), we get:

$$\Delta t_{switch} = 1 + \gamma^2 \left(1 + k \frac{t_{RelObjA}}{t_{RelObjB}}\right)^2$$

Or

$$\Delta t_{switch} = 1 + \left[\gamma \left(1 + k \frac{t_{RelObjA}}{t_{RelObjB}}\right)\right]^2$$
$$= 1 + \left[\gamma + k_1 \frac{t_{RelObjA}}{t_{RelObjB}}\right]^2$$

which is the computed mathematical (model) equation for my proposed switching time between General Relativity time and the Quantum time. Hence, we may construct our fuzzy impulse function by the following procedure through the Maple Soft (that is inspired from its guiding description web site):

1. Piecewise and split the testing time mathematical model equation by the range into two parts, that say:

$$mu\_A := \begin{cases} \dfrac{1}{1 + \left(\dfrac{x}{5} - 5\right)^2} & 25 < x \\ 1 & otherwise \end{cases}$$

2. Plot the set mu_A – the General Relativity time;

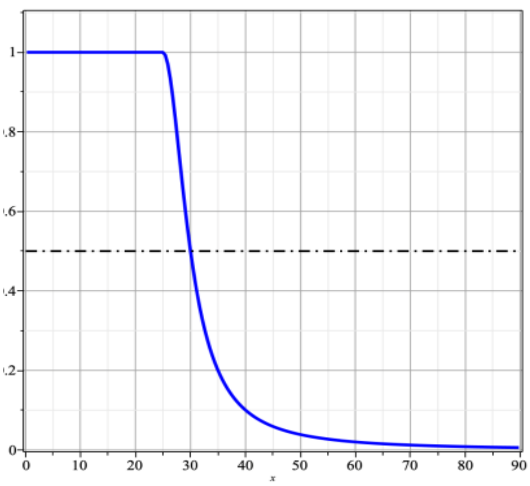


Fig 4.3 Affiliation function of the fuzzy subset A="General Relative Time"

3. Divide the groups into mu_A – the General Relativity time & mu_B – the Universe Absolute time;

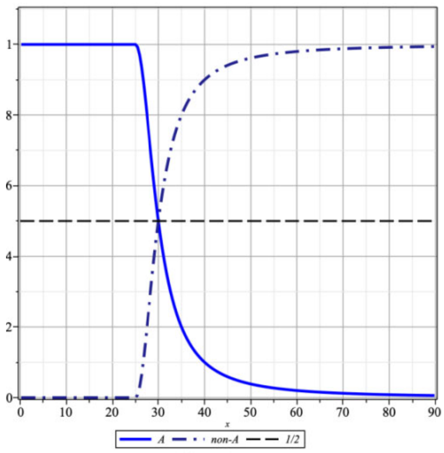


Fig.5 Affiliation functions

4. Transform the testing time model mathematical equation into its mirror inverse image just like below:

$$mu\_A := \begin{cases} 1 & 25 < -x \\ \dfrac{1}{1+\left(-\dfrac{x}{5}+5\right)^2} & \text{otherwise} \end{cases}$$

5. Plot the set mu_A – the General Relativity time again;

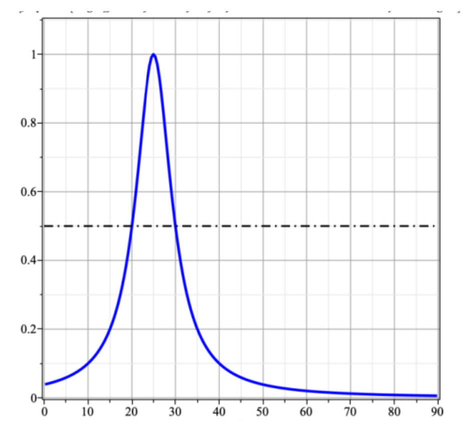


Fig. Affiliation function of the fuzzy subset A="Mirror General Relativity Time Image"

6. Set one more group mu_c for the General Relativity time;

$$mu\_C := \begin{cases} \dfrac{1}{1+\left(-\dfrac{x}{4}+4\right)^2} & -x < 16 \\ 1 & \text{otherwise} \end{cases}$$

7. Plot the corresponding relative graph for the object A and object C

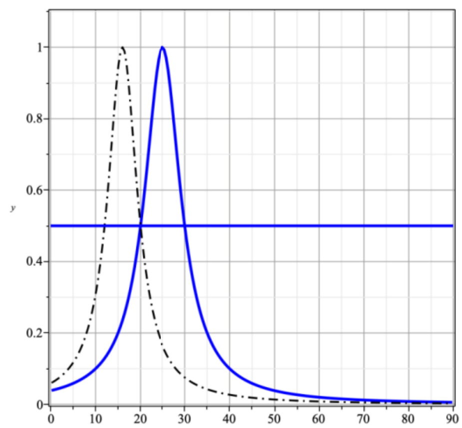


Fig. Affiliation function of the fuzzy subset A="General Relative Time for Object A & Object C"

7.5 Plot all of the related objects' General Relative time

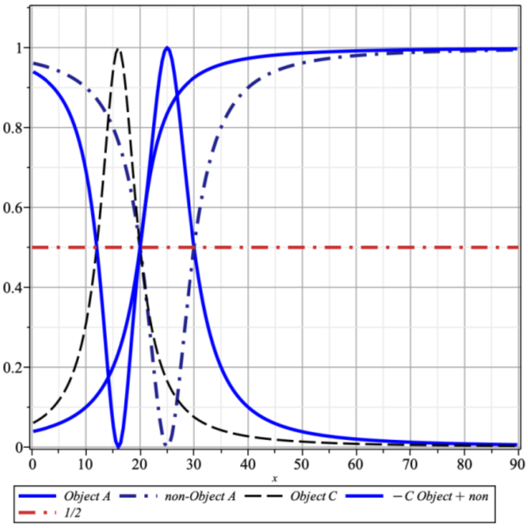


Fig. Affiliation functions

8. Replacing the fuzzy (step) impulse function by my proposed model of the Riemann Zeta Non-Trivial Zeros -- $\frac{cot(x)}{ln(x)}$;

$$mu\_A := \begin{cases} 1 & 25 < -x \\ \dfrac{1}{1+\left(-\dfrac{\cot(x)}{5\ln(x)}+5\right)^2} & otherwise \end{cases}$$

9. Plot the fuzzy (step) impulse function of the group mu_A;

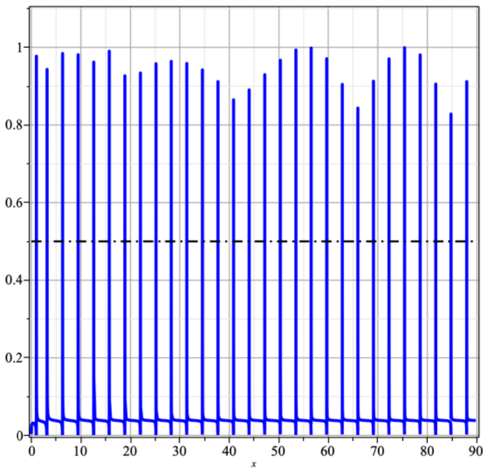


Fig. Affiliation function of the fuzzy subset A="Riemann Zeta Model Fuzzy Step Function"

10. Convert the Riemann Zeta Fuzzy Impulse Function into the Riemann Zeta Fuzzy Fourier Series, we thus obtain:

$$\int_{-\infty}^{\infty} \frac{1}{1+\left(\frac{-cot(x)}{tlog(x)}+t\right)^2} e^{Itx}\, dt$$

or in terms of an infinite summation, we may have:

$$\sum_{x=1}^{\infty} \left| \frac{1}{1+\left(\frac{-cot(x)}{tlog(x)}+t\right)^2} \right| e^{Itx} = \sum_{t=1}^{\infty} |cos^2\theta|\, e^{Itx} \qquad \text{where } \tan\theta = \left(\frac{-cot(x)}{tlog(x)}+t\right)$$

for the wanted and expected engineering sense of digital signal processing.

Or $\sum_{x=1}^{\infty} |cos^2\theta|\, (cos 14.1317x + Isin 14.1317x)$

i.e. A complete Fuzzy Fourier Series Equation (that say 0.5+14.1317I) with a Taylor Expansion Substitution To Approximate:$\sum_{x=1}^{\infty}\left(1+\left(\frac{-cot(x)}{14.1317log(x)}+14.1317\right)^2\right)(cos14.1317x)+\sum_{x=1}^{\infty}\left(1+\left(\frac{-cot(x)}{14.1317log(x)}+14.1317\right)^2\right)(Isin14.1317x)$

i.e.

$$\sum_{x=1}^{\infty}\left(1+\frac{200.0384174*x^2}{(x-1.)^2}-\frac{399.7432231*x}{(x-1.)^2}+\frac{198.7041102}{((x-1.)^2)}+\frac{1.000000000}{(x*(x-1.)^2)}+\frac{0.001251846820}{(x^2*(x-1.)^2)}\right)(cos14.1317x)$$

+

$$\sum_{x=1}^{\infty}\left(1+\frac{200.0384174*x^2}{(x-1.)^2}-\frac{399.7432231*x}{(x-1.)^2}+\frac{198.7041102}{((x-1.)^2)}+\frac{1.000000000}{(x*(x-1.)^2)}+\frac{0.001251846820}{(x^2*(x-1.)^2)}\right)(Isin14.1317x)$$

Or in general, we have:

$f(x) = a_0 + \sum_1^{\infty} a_n\, cosnx + \sum_1^{\infty} I\, b_n sinnx$

11. Apply the Taylor Series Expansion to the above Fuzzy Fourier Series, we obtain the following expression as shown below:

$\left(\sum_{x=1}^{\infty} 1-\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+O(x)^8\right)^2 e^{2Itx}$ where $\cos\theta = \sqrt{1+\left(\frac{-cot(x)}{tlog(x)}+t\right)^2}$

12. Plotting the Fuzzy Fourier Series with the Taylor Series Expression:

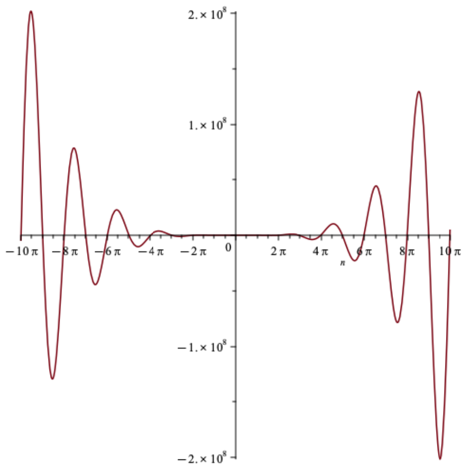


Figure: The paired Damped Harmonic Oscillated curve is related to the Conformal Quantum Field Theory and in the area of condensed matter with advanced materials for the sequence {a_n}.

The actual integrated Fuzzy Fourier Series Model Equation should be:

a_n=$(-98.5724n+242.677n^3)cos[n[\pi]]+(31.3766-180.471n^2+253.212n^4)sin[n[\pi]]$
$= a'_n cos\frac{(n\pi)}{L}$
b_n=$(-154.321n+253.211n^3)cos[n[\pi]]+(49.1219-242.204n^2)sin[n[\pi]]$
$= b'_n sin\frac{(n\pi)}{L}$

where the fuzzy Fourier series: $f_1(x)=\sum_{n=1}^{\infty}\left(a'_n cos\frac{(n\pi)}{L}+b'_n sin\frac{(n\pi)}{L}\right)$ with L = 1.

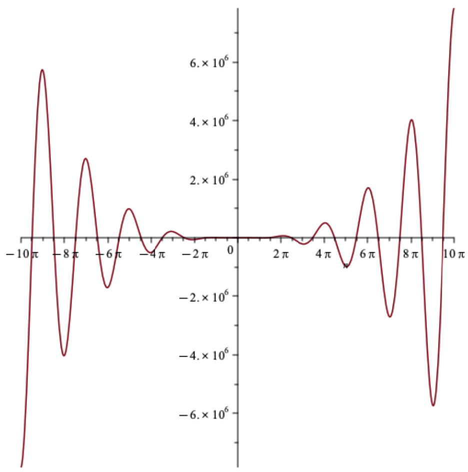


Figure: The plotting of the Fuzzy Fourier Sequence {b $_n$}.

Or in terms of the Fourier Series Language, we have:

$$g(t) = \frac{1}{2\pi}\int_{-1}^{1}\big(f(x)\big)\, e^{itx}dx = \frac{\big((-98.5724t+242.677t^3)cos[t[\pi]]+(31.3766-180.471t^2+253.212t^4)sin[t[\pi]]\big)t}{t^5(2t\pi+sin(2t\pi))} + \frac{\big((-154.321t+253.211t^3)cos[t[\pi]]+(49.1219-242.204t^2)sin[t[\pi]]\big)t}{t^4(2t\pi-sin(2t\pi))}$$

$$f(x) = \int_{-1}^{1}\big(g(t)\big)\, e^{-itx}dt = (x^2)x - 1^2+200.0384174x^4 - 399.7432231x^3+198.7041102x^2+1.00000000x+0.001251846820$$

which is just the Fourier Analysis Paired Functions according to the Fourier's Integral theorem in [Principle of Quantum Mechanics, p.26-27].

N.B. The full Fuzzy Fourier Series at the first Riemann Non-Trivial Zeta Zeros, i.e. when t = 14.1317, should be:

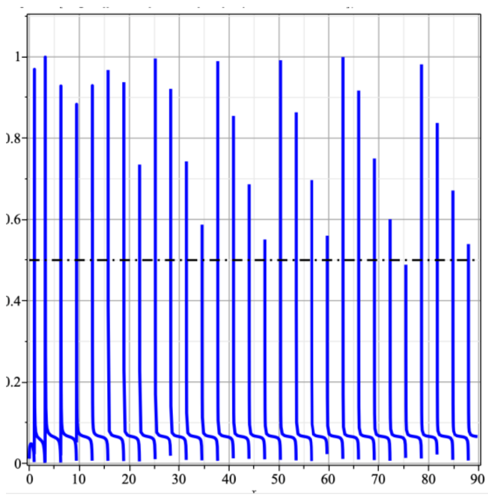
$$\left(\sum_{x=1}^{\infty} 1-\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+O(x)^8\right)^2 e^{2Itx}$$

where $\cos\theta = \sqrt{1+\left(\frac{-cot(x)}{14.1317log(x)}+14.1317\right)^2}$ or $\cos\theta = -\frac{-x^2+3}{14.1317*3x}\frac{x+1}{2(x-1)}+14.1317$.

In practice, we just have:

$$mu\_A := \begin{cases} 1 & 16 < -x \\ \dfrac{1}{1+\left(-\dfrac{0.2660133379\cot(x)}{\ln(x)}+3.759220687\right)^2} & otherwise \end{cases}$$

mu_A:=piecewise(16< − x, 1, 1/(1+((−cot (x)/ln (x)+14.1317)/3.75921)^2))



N.B. According to the fact that, $\sum_{-\infty}^{\infty}\frac{1}{1+n^2} = \int_{-\infty}^{\infty}\frac{1}{1+x^2}dx = \pi\coth\pi$ or we have successfully regularized the prescribed discrete Fuzzy Fourier Series.

In practice, according to the general form of the (high/low) filter equation, we may have: $H(s)=\frac{1}{1+\tau s} = \frac{1}{1+x^2}$ with also $H(s)=\frac{1}{1+\tau s} = \frac{\omega_c}{s+\omega_c}$
where $\omega_c=\frac{1}{\tau}$, i.e. In the present case, $\tau=s=x$ or $\omega_c=\frac{1}{x}$ which is also an impulse response function and hence it will also constitute another filter. In such case, this is just the "Filter of the Fuzzy Filter".

N.B. For reader's interest, if we convert the Fuzzy (Step) Impulse function into a Heaviside function, we will get:

$$f := 1 - \text{Heaviside}(x+25) + \frac{\text{Heaviside}(x+25)}{26+\dfrac{\cot(x)^2}{25\ln(x)^2}-\dfrac{2\cot(x)}{\ln(x)}}$$

Then we may apply the Fourier transform for the later part of the expression to it to get:

$$
\begin{aligned}
&-\frac{5\, e^{-\frac{\mathrm{Abs}[w]}{5\sqrt{2}\sqrt{-\frac{\cot}{\cot+650\ln}}}}\ln\sqrt{-\frac{\cot}{\cot+650\ln}}\sqrt{\pi}}{4\cot}-\frac{i\, e^{-25 i w}}{\sqrt{2\pi}\, w}+\\
&\frac{5\, e^{-\frac{i\sqrt{\cot+650\ln}\, w}{5\sqrt{2}\sqrt{\cot}}}\ln \mathrm{CosIntegral}\left[-\frac{\sqrt{\cot+650\ln}\, w}{5\sqrt{2}\sqrt{\cot}}\right]}{4\sqrt{\cot}\sqrt{\cot+650\ln}\sqrt{\pi}}-\frac{5\, e^{\frac{i\sqrt{\cot+650\ln}\, w}{5\sqrt{2}\sqrt{\cot}}}\ln \mathrm{CosIntegral}\left[\frac{\sqrt{\cot+650\ln}\, w}{5\sqrt{2}\sqrt{\cot}}\right]}{4\sqrt{\cot}\sqrt{\cot+650\ln}\sqrt{\pi}}-\\
&\frac{5\, e^{-\frac{i\sqrt{\cot+650\ln}\, w}{5\sqrt{2}\sqrt{\cot}}}\ln \mathrm{CosIntegral}\left[\left(25-\frac{\sqrt{\cot+650\ln}}{5\sqrt{2}\sqrt{\cot}}\right)w\right]}{4\sqrt{\cot}\sqrt{\cot+650\ln}\sqrt{\pi}}+\\
&\frac{5\, e^{\frac{i\sqrt{\cot+650\ln}\, w}{5\sqrt{2}\sqrt{\cot}}}\ln \mathrm{CosIntegral}\left[\left(25+\frac{\sqrt{\cot+650\ln}}{5\sqrt{2}\sqrt{\cot}}\right)w\right]}{4\sqrt{\cot}\sqrt{\cot+650\ln}\sqrt{\pi}}-\sqrt{\frac{\pi}{2}}\,\mathrm{DiracDelta}[w]+\sqrt{2\pi}\,\mathrm{DiracDelta}[w]-\\
&\frac{5\, i\ln\sqrt{-\frac{\cot}{\cot+650\ln}}\, w\, \mathrm{MeijerG}\left[\left\{\left\{\frac{1}{2}\right\},\{\}\right\},\left\{\left\{\frac{1}{2},\frac{1}{2}\right\},\{0\}\right\},-\frac{(\cot+650\ln)\, w^2}{200\cot}\right]}{4\cot \mathrm{Abs}[w]}+\\
&\frac{5\, i\, e^{-\frac{i\sqrt{\cot+650\ln}\, w}{5\sqrt{2}\sqrt{\cot}}}\ln \mathrm{SinIntegral}\left[\frac{\sqrt{\cot+650\ln}\, w}{5\sqrt{2}\sqrt{\cot}}\right]}{4\sqrt{\cot}\sqrt{\cot+650\ln}\sqrt{\pi}}+\frac{5\, i\, e^{\frac{i\sqrt{\cot+650\ln}\, w}{5\sqrt{2}\sqrt{\cot}}}\ln \mathrm{SinIntegral}\left[\frac{\sqrt{\cot+650\ln}\, w}{5\sqrt{2}\sqrt{\cot}}\right]}{4\sqrt{\cot}\sqrt{\cot+650\ln}\sqrt{\pi}}+\\
&\frac{5\, i\, e^{-\frac{i\sqrt{\cot+650\ln}\, w}{5\sqrt{2}\sqrt{\cot}}}\ln \mathrm{SinIntegral}\left[\left(25-\frac{\sqrt{\cot+650\ln}}{5\sqrt{2}\sqrt{\cot}}\right)w\right]}{4\sqrt{\cot}\sqrt{\cot+650\ln}\sqrt{\pi}}-\\
&\frac{5\, i\, e^{\frac{i\sqrt{\cot+650\ln}\, w}{5\sqrt{2}\sqrt{\cot}}}\ln \mathrm{SinIntegral}\left[\left(25+\frac{\sqrt{\cot+650\ln}}{5\sqrt{2}\sqrt{\cot}}\right)w\right]}{4\sqrt{\cot}\sqrt{\cot+650\ln}\sqrt{\pi}}
\end{aligned}
$$

Figure: An investigation to the frequency or time domain of my proposed Fuzzy Fourier Series by a Fourier Transform. In fact, we may encode the secret data as described in the aforementioned way. Hence, we may decode through the reverse of the above prescribed process.

Actually, if we try to optimize the finalized outcome of the Fuzzy Fourier Series such as the case below:

Optimization[Maximize](1.273239544*10^(-11)*(1.309429455*10^15*(-1)^n*n^4 - 7.578969782*10^14*n*(-1)^n)/n^5, n = -100 .. 100)

We may obtain:

[1667.09449891235, [n = 9.98987870354932]]

and

Optimization[Maximize](6.366197722*10^(-8)*(1.245341994*10^11*(-1)^n*n^3 - 7.572681350*10^10*n*(-1)^n)/n^4, n = -100 .. 100)

We may obtain:

[788.378320735371, [n = 9.98998538154174]]

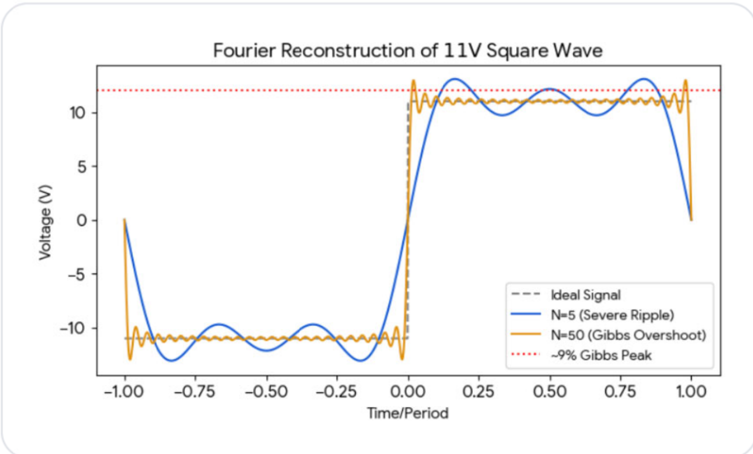
In brief, the most common application of the Fourier series is employed in the field of engineering sense like the digital signal filtering system for the imagine or sound processing. Hence, my proposed fuzzy Fourier series may be used in the establishment of the digital fuzzy signal system which is different from the traditional one that the fuzzy can have human-made artificial intelligent to engineering identify the most suitable image signal for a restoration while the others may be saved for some alternative options or purposes. Thus, the outcome processed image will be more sharped and cleared than the classical digital signal filter. Practically, in order to bridge the time gap between the quantum mechanics and the general relativity, we may need to "synchronize" their corresponding time. In the engineering sense, this fact may be referred to the "Predict the Chaotic Time Series

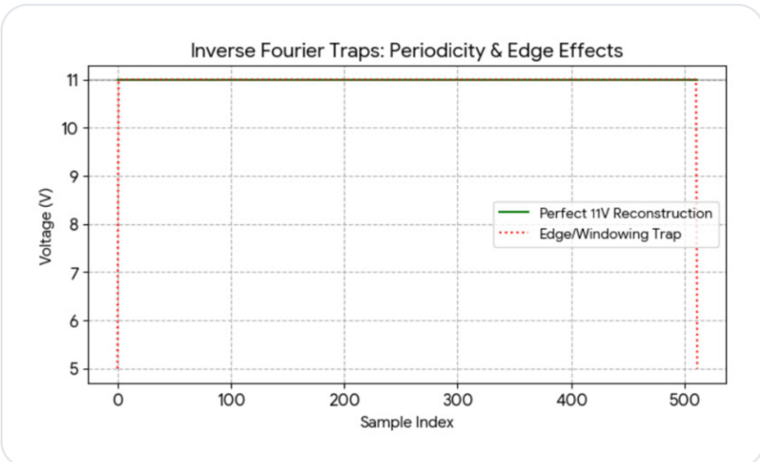
Using Type-2 FIS" as suggested by the U.S.A. Matlab's Japanese reference web site.In fact, we may need to start the "gravitise" the quantum mechanics together with the present trials of quantize the gravity in both ways for the determination about the feasibilities of quantum gravity (whether their meeting point may be what we were most wanted) or else any other alternatives just like the case of multi-dimension or even a multiverse. Certainly, one may further develop my Fuzzy Fourier Series into something like the A.I. chips in the sense of engineering for a wide and in-depth study. However, this writer is just a private researcher and has no such private & infinite resources to do. The role of the recent paper is to act as a starter for the beginning research in the (Quantum Fuzzy Bayesian Optimized) A.I. chips while one may need the Western countries such as the United States, England or the European Union & Japan's large University academic research centers for the continuing of my present research.

(N.B. There is always a mirror imaged inverse Fuzzy Fourier Series like the following:

$$\tilde{f}(x)=\frac{\tilde{a}_0}{2} \oplus \sum_{n=1}^{N} \left[\tilde{a}_n \otimes \cos\frac{n\pi x}{L} \oplus \tilde{b}_n \otimes \sin\frac{n\pi x}{L}\right]$$

This author wants to remark that there are always trap(s) in the fourier series because of the Gibbs Phenomenon etc. The result may be the unexpected excessive amount of the overshoot voltage that may fry the electronic hardware such as the MOSEFT. In order to solve such problem, one may need to add 10% headroom or windowing functions like Hamming or Hann. In the case of simulation, one may need to use Lanczos Sigma smoothing etc. By the way, in the mirror inverse or the vice versa way, from the level of excessive voltage, one may estimate the amount of the "traps" that appears on the Fourier series and restore its mathematical equations. Certainly, there are still rooms of research (or the research gaps found from the web) in such of the above Fourier series trap topic but the focus of the present paper is in the study in the revolutionized formulation of a novel Fuzzy Fourier Series from the model of the Riemann non-trivial zeta function.)



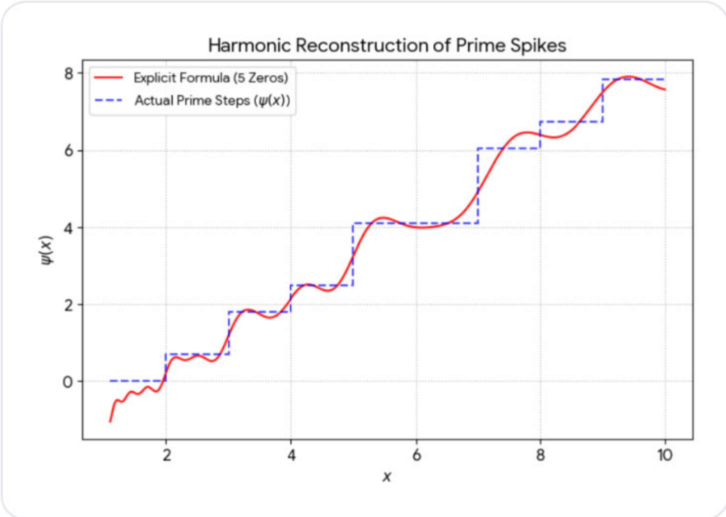


(N.B. In practice, the Fourier Series and its (vice versa or mirror imaged inverse) Fourier Series constitutes a sandwiched pair which may squeeze both of the voltage and the so-called Fourier "Traps" which may be as a result used to model those traps. However, as a private research, this author leaks of experimental appliances for the continuation in the such study and thus this author will leave such study to other research institute in the university.)

(N.B. In reality, the Fourier analysis and Fourier synthesis can sandwich those "Traps" because:

1. Fourier series can "trap" those of the sharp signal jumps which is mathematically proven to converge to the exact midpoint of the jump. That is it "sandwiches" the gap by setting right in the middle of it.

2. In fact, one may consider those high pitched hum as the "traps". With the sandwiching between the forward and inverse transform, all of the specific high-frequency spike will then be removed, then one may rebuild the signal without the hum.)



As shown in the graph, the "sandwiching" effect is the red wave oscillating around the blue "staircase." The more zeros you include, the tighter the red line clings to the steps.

A Comparison of Riemann Zeta Non-Trivial Zeros Error Estimation between my proposed Taylor Approximation and the standard contour way of approximation

In the present section, this writer will try his every effort to compare the error estimated between my proposed Taylor Approximation and the standard contour way of approximation to the critical line x = 0.5 for those of the non-trivial zeta zeros. In fact, similar argument in the Fourier Analysis Pair does apply in the case between the Riemann Zeta non-trivial zeros and the Prime Number Distribution. With reference to the British Exeter University, we may have the following Riemann Weil Explicit Formula:

$$\sum_{\gamma} h(\gamma)= h\left(\frac{i}{2}\right)+h\left(-\frac{i}{2}\right) - g(0)log\pi+\frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{1}{2}ir\right)dr - 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}g(logn)$$

where $h(r)=\int_{-\infty}^{\infty} g(u)e^{iru}du$ is obviously a Fourier Analysis pair of $g(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)e^{-iru}dr$. Hence, the Fourier transform of the Prime Number Theorem's Error can be used to estimate the corresponding errors that associated non-trivial zeta zeros within the critical line x = 0.5 when we are comparing with the other lines of x in the critical strip region $x \in (0,1)$. That say, the Fourier transform to PNTE is used to determine whether the critical line at x = 0.5 is the most optimized one (with the best fitted non-trivial zeros) among the critical strip region with other feasible lines $x \in (0,1)$where $x \neq 0.5$.

In reality, according to the [Prime Number with Error Term, MIT, p.36-37], the Perron's formula is equivalent to:

$$\left|\sum_{n<x}\frac{f(n)}{n_0^S}-\frac{1}{2\pi I}\int_{b-IT}^{b+IT}F(s_0+s)\frac{x^s}{s}ds\right| \leq \frac{10x^{bB}(\sigma_0+b)}{T}+100*2^{b+\sigma_0}x^{1-\sigma_0}H(2x)\frac{logx}{T}$$

In practice, if we take F(s) = $\frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}$ with $\sigma_0 < 1$

H(n) = log n, $B(\sigma) = \frac{10}{\sigma-1}$, $s_0 = 0$, $b = 1+\frac{1}{logx}$, amend the above bounding equation a little bit and substitute back into the Maple Soft for the optimization in the subject "u", then we may get:

$$f{:=}10*x^{\left(1+\frac{1}{(log*x)}\right)}*\left(\frac{10}{\left(u+\frac{1}{(log*x)}\right)}+\left(100*2^{\left(1+\frac{1}{(log*x)}+u\right)}*x^{(1-u)}*log(2*x)\right)*2*log*x\right)$$

Optimization[Maximize](f, u = -1 .. 1);
[[1.134038940016870400*10^{18} , [log = 766.763990951222,
u = 0.425057630472206, x = 11323.2965161185]]

Optimization[Maximize](f, u = -1.1 .. 1);
[[1.333112805391003648*10^{18}, [log = 724.001969065715
u = 0.382714325653233, x = 10727.8199514885]]

Optimization[Maximize](f, u = -0.98 .. 1);
[[1.098541833791412608*10^{18} , [log = 775.930517953597, u = 0.433525181717510, x = 11449.2221372478]]

As there is a change of value of u from 0.425 to the value u = 0.3827 until u = 0.433525, we may conclude that there is a local minimum at
u = 0.3827. Now let's consider the following equation:

$$f := \left(5 * x^{\left(1+\frac{1}{(log*x)}\right)}\right) * \left(\frac{10}{\left(u+\frac{1}{(log*x)}\right)}\right) + \left(100 * 2^{\left(1+\frac{1}{(log*x)}+u\right)} * x^{(1-u)} * log(2*x)\right) * 0.5 * log * x$$

Optimization[Maximize](f, u = -1 .. 0.7);
[[[5.77373559861951561*10^{81}, [log = 2.47267773971619*10^{18},
u = -1.,]]x = 9.99684243438673*10^{19}]]

Optimization[Maximize](f, u = -1 .. 0.6);
[[[7.22151334895720538*10^{82}, [log = 5.90412547172041*10^{19} , u = -1.,
]] x = 8.07089229417555 10^{19}]]

Optimization[Maximize](f, u = -1 .. 0.5);
[[[3.47390867039431157*10^{82}, [log = 8.23388797516893*10^{19}, u = -1.,]]
x = 5.67460032269837*10^{19}]]

Optimization[Maximize](f, u = -1.. 0.4); [[
[3.85524259400554315*10^{82}, [log = 8.06333970338030*10^{19}, u = -1.,]]
x = 5.91443596954865 10^{19}]]

Optimization[Maximize](f, u = -1 .. 0.3); [[
[4.29069049822857044*10^{82}, [log = 7.86240474225649 10^{19}, u = -1.,]]
x = 6.17904456036228*10^{19}]]

Obviously, there is a maximum at u = 0.5 & 0.6 as the value of f attains its maximum at 7.22151334895720538*10^{82} with a change from 5.77373559861951561*10^{81} to 3.47390867039431157*10^{82}.

In brief, for the Prime Number Theorem Error Equation, we have to minimize:

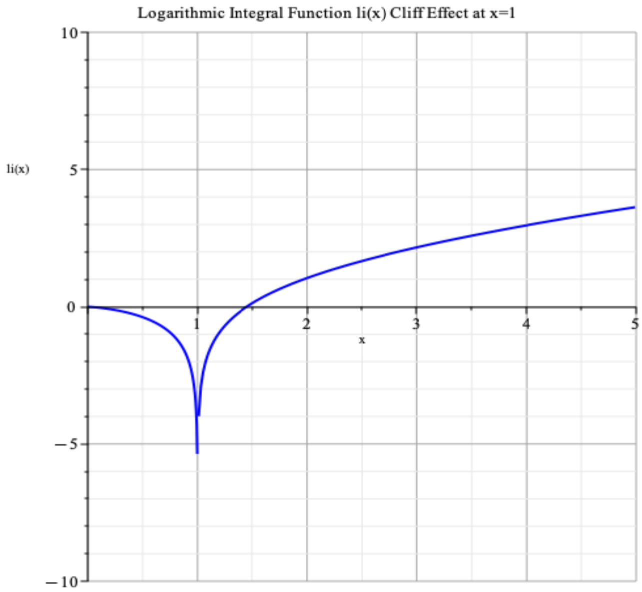
$$\frac{\zeta'}{\zeta} + \frac{1}{s-1} = \int_1^{\infty}\left(x - \psi(x)\right) x^{-s-1} dx$$

Actually, when the Perron's formula attains its maximum, ψ(x) will also attain its maximum, then the equation $\int_1^{\infty}\left(x - \psi(x)\right) x^{-s-1} dx$ will attain its minimum at x = 0.5 or the critical strip in the critical region $x \in (0,1)$. Hence, this writer conclude that the Riemann Hypothesis must be correct or the critical strip x = 0.5 is just the most optimized line with most of the non-trivial zeros. In fact, from my opinion, there are also other non-trivial zeros stay all around the critical region as there are still $\Re\left(\zeta(x+It)\right)$ meets $\Im\left(\zeta(x+It)\right)$, but their differences with the $\Re\left(\zeta(0.5+It)\right)$ are only lying in their optimizations or a shift of the horizontal axis $y_h = 0$. That says, for $\Re\left(\zeta(0.5+It)\right)$ meets $\Im\left(\zeta(0.5+It)\right)$ at the normal horizontal axis or $V_h = 0$ (with a 3-dimensional sense, Re, Im & V_h. Thus, if we can shift the horizontal axis V_h with a delta δ, some others non-trivial zeros will still appears but it is NOT on the critical line x = 0.5. That is, these "(abnormal) non-trivial zeros" are staying outside the critical line x = 0.5 with the following conditions are satisfied:

$V_h = \Re\left(\zeta(0.5 \pm \Delta+It)\right) = \delta = \Im\left(\zeta(0.5+\Delta+It)\right)$
I.e. $V_h = \Re\left(\zeta(0.5 \pm \Delta+It)\right) - \delta = 0 = \Re\left(\zeta(0.5+It)\right)$ for any given $\Delta \geq 0$.
Or for one of the case x = 0.7 when $V_h = \Re\left(\zeta(0.5+0.2+It)\right) - \Re\left(\zeta(0.5+It)\right) - \delta_1 = \Im\left(\zeta(0.5+0.2+It)\right) - \Im\left(\zeta(0.5+It)\right) = 0$ etc.

DISCUSSIONS & COMMERCIAL APPLICATIONS

In practice, by considering the Taylor approximation for the function $\frac{1}{lnx}$ around x = 1, we may get a sudden negative infinity drop (or the cliff effect) which is just like the following figure:



The Maple series expansion for the Li(x) function is:

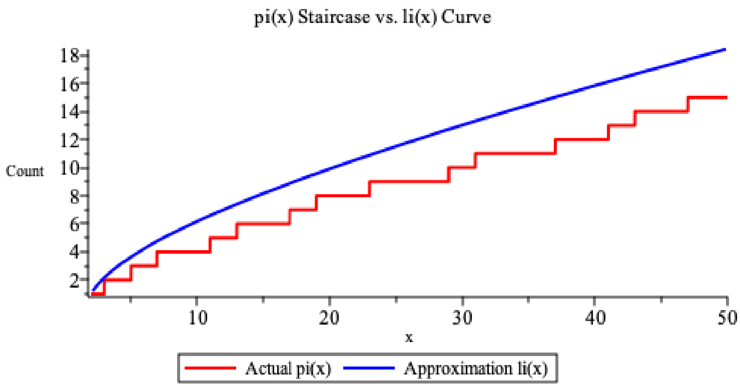
$$\frac{1}{ln\mu}(x-\mu)-\frac{1}{ln(\mu)^2}(x-\mu)^2+\ldots$$

Near the cliff (at x = 1),

$$li(1+\varepsilon)\approx\gamma+ln|ln(1+\varepsilon)|+\sum\frac{(ln(1+\varepsilon))^n}{n.n!}$$

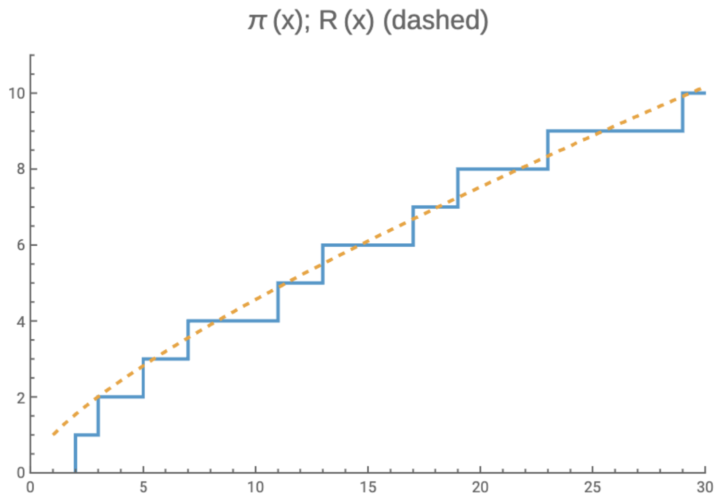
A Comparison between Li(x) and $\pi(x)$

x	Actual $\pi(x)$	li(x) Principal Value	Difference (li(x)-$\pi(x)$)
2	1	1.045	+0.045
10	4	5.120	+1.120
50	15	16.31	+1.31
100	25	29.08	+4.08
1,000	168	176.56	+8.56



A Comparison between Perron's formula and $\pi(x)$:

x	Actual $\pi(x)$	Perron's Main Term (Li(x))	Perron's +10 zeros	Numerical Difference (smooth)
2	1	1.05	1.25	+0.05
10	4	5.12	4.48	+1.12
50	15	16.08	15.31	+1.08
100	25	29.08	26.15	+4.08
1000	168	176.56	171.22	+8.56



A Comparison between Perron's formula, li(x) and $\pi(x)$:

Actual $\pi(x)$	li(x) Principal Value	Perron's Main Term (li(x))	li(x)-$\pi(x)$	Perr(x)-$\pi(x)$	li(x)-Perr(x)
1	1.045	1.05	+0.045	+0.05	0.04
4	5.120	5.12	+1.120	+1.12	0
15	16.31	16.08	+1.31	+1.08	0.23
25	29.08	29.08	+4.08	+4.08	0
168	176.56	176.56	+8.56	+8.56	0

From the above tables, one may observe that Perron's formula is a good approximation to the Li(x) function while there may be some differences between two are zeros. In fact, other than the Prime Number Theorem, there is also a Green Tao Theorem which is just the result of the Inverse Prime Number Theorem. To be precise, what one of the Green Tao theorem's research gap is the problem of "Prime Density". By the way, this writer may self-develop a method to determine such kind of density like the following:

$$\frac{1}{1+lnk}<ln(k+1)<\sum 1/n<1+lnk<\frac{1}{ln(k+1)}$$
$$\frac{1}{1+lnk}-lnk<ln(k+1)-lnk<\sum 1/n-lnk<1+lnk-lnk<\frac{1}{ln(k+1)}-lnk$$
$$\frac{1}{1+lnk}-[(lnk)+1]<\frac{1}{1+lnk}-lnk<ln(k+1)-lnk<\sum 1/n-lnk<1+lnk-lnk<\frac{1}{ln(k+1)}-lnk<\frac{1}{ln(k)}-(lnk)$$

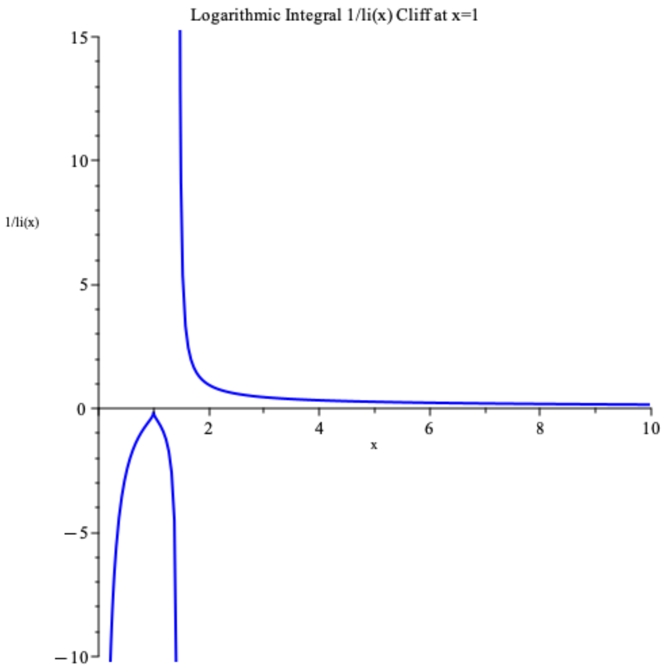
By the Taylor Expansion series, $\frac{1}{lnk}-lnk \approx -\frac{2}{e}(k-e)+\frac{2}{e^2}(k-e)^2-\frac{8}{3e^3}(k-e)^3$

The upper bound for the above Taylor Series is 0.0717 while the error is guaranteed to be 0.00000717 and the maximum value M is just 1.7216.

The lower bound for the above Taylor Series is 0.0368. Hence the optimal symmetric error is ± 0.05425 which implies the series will be converged.

As the limit when k $\rightarrow \infty$, $\sum 1/n - lnk \rightarrow \gamma$ = 0.577, therefore $\frac{1}{lnk}$ is bounded above by 1.7216 – 0.577 = 1.1446 which is just this writer's proposed prime density.

The implication is that one may apply the Szemerèdi's Theorem directly to guarantee arithmetic progression. In reality, the "Inverse Green-Tao Theorem" generally refers to the Inverse Theorem for the Gowers Norms which is a critical tool used by Ben Green and Terrence Tao to complete the proof of their famous result on arithmetic progression in primes. As the prime density is approximately equals to 1.1446, this shows that the primes start to behave like a pseudorandom measure which is required for the Green-Tao Theorem. Accordingly, the W-tricked primes are not correlated with nilsequences, the inverse Green-Tao Theorem implies their Gowers norms must be small. Then the primes are officially "pseudorandom" enough to apply the Szemerèdi's Theorem and guarantee the primes contain the same density of arithmetic progression just like the perfect random set of the same size.



But on the other hand, we may also consider such fall as an impulse response h(t) of a control system where h(t) is the inverse Laplace transform of the proposed control system with the transfer function H(s), i.e. h(t) = L^{-1}(H(t)). Therefore the transfer function H(s) of the such suggested engineering control system should be the Laplace transform of the response function h(t). Or

L(h(t)) = H(t)

i.e. H(t) = L($\frac{1}{lnt}$) = L($(lnt)^{-1}$).

But as $\frac{1}{lnt}$ = ∞ when t = 0 or 1 and otherwise are a constant c. Then we may define a Dirac delta function as follow:

$$\frac{1}{c}\text{Dirac}(\frac{1}{lnt}) = \begin{cases} \infty & \text{when } t = 0 \\ 1 & \text{otherwise} \end{cases}$$

which is just the inverse Laplace transform of the engineering control system H(t). Thus,

H(t) = L($\frac{1}{c}$Dirac($\frac{1}{lnt}$))

= L($\frac{1}{c}$)

=$\frac{1}{c}$*($\frac{1}{s}$)

=$\frac{1}{c}$Dirac(s)

If we further replace t by (1-t), then H(1-t) = $\frac{1}{c}*\left(\sum_{n=0}^{\infty} C_n^p(-1)^n s^n\right)$.

Alternatively, we may let y = ln(t), then ln y = ln $(\ln(t))^{-1}$ = -ln (ln(t)) or
ln (ln(t)) = -ln(y) and we have:

L($-ln(lnt)$) = L($-ln(y)$)

= $\frac{1}{s}[\gamma - ln(s)]$

$-ln(lnt)$= L^{-1}($\frac{1}{s}[\gamma - ln(s)]$)

$(lnt)^{-1}$= $e^{\frac{p(\gamma-ln(t))}{t}}$

L($(lnt)^{-1}$) = L($e^{\frac{p(\gamma-ln(t))}{t}}$)

= $\frac{t^{-p}*e^{p\gamma}}{p}$

(N.B. One may express the above Laplace transform in terms of a Dirac function.)

If we further replace t with 1 – t and expand the Laplace transform with Taylor series, then we may get:

H(1-t) = L($\left(ln(1-t)\right)^{-1}$) = $\left(\sum_{n=0}^{\infty} C_n^p(-1)^n t^n\right)*\frac{e^{p\gamma}}{p}$

which is just the expected commercial engineering control system transfer function with an impulse response h(1-t) = $\frac{1}{ln(1-t)}$.

There are also some other commercial engineering usage in both of the Black Hole Control System and so as the respective Inverse Control System. However, such kind of research may be too width and too depth as the present study is just a private one where this author cannot afford and will be left to those interested parties. In brief, we may consider the singularity (if really exists) of the black hole as a kind of engineering impulse and start our research of using black hole as a control system or even as a type of a quantum computing etc. To be precise, we may generally have our so-called "Black-Hole Engineering".

Furthermore, consider the following Mathematica scripting code for the boundary of the Dirichlet Condition, we may get:

DSolveValue[{$\nabla^2_{x,y} U[x,y]$ == 0,
DirichletCondition[u[x, y] == Sin[4 ArcTan[x, y]], True]},
u[x, y], {x, y} \[Element] Disk[]]

Actually, the above computation may be used in the inverse Fourier transform (or in fact the complex contour integral) for the boundary condition of solving Schrodinger Equation so as to obtain the machine learning operator model for the non-trivial Zeta Zeros. In practice, one may employ the following steps (or algorithm) to obtain the expected operator model just like:

A. From the true values of the non trivial Zeta Zeros to get a finite operator model;
B. From the finite operator model to get an infinite operator model;
C. From the infinite operator model to get a regression equation model;
D. From the regression equation model and the representer theorem to get the wanted Hilbert Space;
E. Modify, improve and verify the operator model with those boundary values.

Conclusions

In brief, with reference to my previous paper in the complex analytic topology, we may apply such prescribed theories and ideas to solve the mysterious Riemann Hypothesis. In practice, what this writer means is that with the subject of the real analysis or the complex analytic topology, we can show the continuity of the function z = *f*(x): [0,1] → ℝ

z = *f*(x) = Re(ξ(x+I y)) – Im (ξ(x+I y)).

By the Fundamental Theorem of Calculus, we may we define the integral function of z as:

Z = *F*(x) = $\int_0^1 \frac{\partial z}{\partial x} \partial x = \int_0^1 \Re\left(\zeta(x+Iy)\right)\partial x - \int_0^1 \Im\left(\zeta(x+Iy)\right)\partial x$
with $\frac{\partial z}{\partial x}$ is just equal to this writer's predefined z = *f*(x) as mentioned in the above or $z = \Re\big(\zeta(x+It)\big) - \Im\big(\zeta(x+It)\big)$ or z = $\frac{\partial z}{\partial x}$ = *F*'(x) = *f*(x).

In fact, according to the computer plotting graphs fig 1,fig 2, fig3, we find that such function of the *f*(x) will attain its zero at x = 0.5 or this is just the situation of the fig 2. Actually, from the theory of the real analysis, in a closed and bounded region likes the critical strip $x \in [0,1]$, there must exist an optimal point, named "c" such that *F*'(c) = *f*(c) = 0. In practice, as shown by the fig 2, such optimal point must be x = 0.5. This is a completion of another alternative proof (by the subject of the real analysis or the complex analytic topology) to determine the truth-less of the long time struggling Riemann Hypothesis. Therefore, this writer has the confident to conclude that the Riemann Hypothesis must be correct.

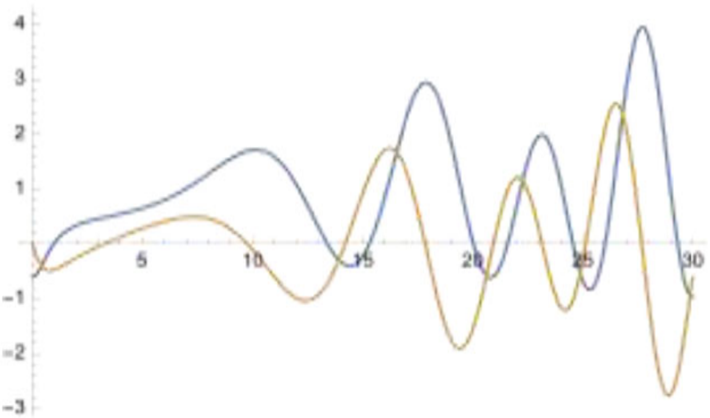


Figure 1. A plot of the Real and Imaginary parts of the Riemann Zeta function at x = 0.1 which has infinite many intersection points lying below the x-axis in the above figure

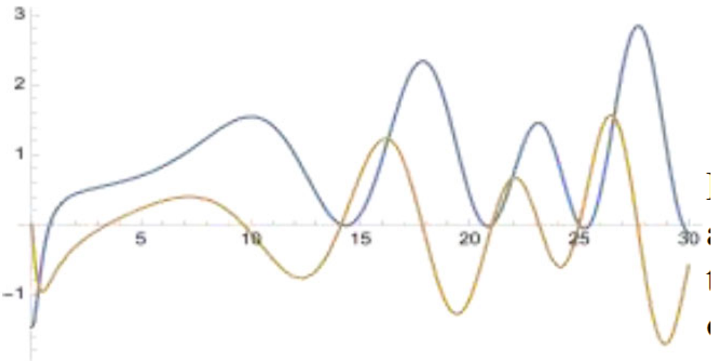


Figure 2. A plot of the Real and Imaginary parts of the Riemann Zeta function at x = 0.5 which has infinite many intersection points at the x-axis in the above figure or at the critical line

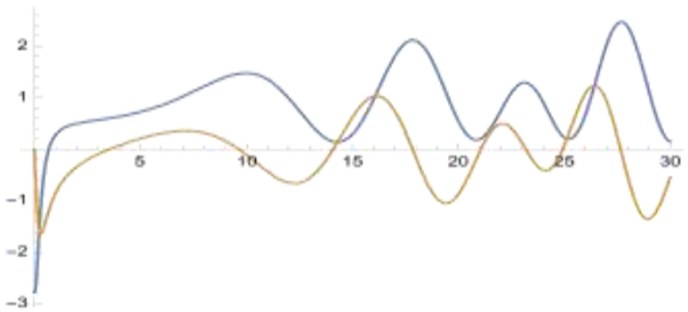


Figure 3. A plot of the Real and Imaginary parts of the Riemann Zeta function at x = 0.7 which has infinite many intersection points lying above the x-axis as shown in the figure

In a nutshell, from the above three figures and my previous, the function f(x) is just a continuous function which changes the values from a negative value of $Z'_{0.1} = F'(x_{0.1})$ at x = 0.1 to a positive value of $Z'_{0.7} = F'(x_{0.7})$ at x = 0.7. Hence, according to the theory of the real analysis in the first derivative test: For a point in a closed and bounded region like the $x \in [0,1]$,with the changing sign from negative (positive) to positive (negative), there must be one optimal point, say "c", somewhere between 0 and 1 such that $F'(c) = f(c) = 0$ (w.r.t. Bolzano's theorem that implies the Immediate Value Theorem where F(c) is just the optimal (or the local minimum / maximum) value (according to the first derivative test). Or with reference to the present situation in the figure 2, this is just the situation for the critical line at c = x = 0.5 which attains its saddle point for the function F(x) (or an optimum in the sense of the function F(x)) and is also with the infinite many roots of f(x) (with the infinite many intersections for z = f(x) = Re(ξ(x+I y)) – Im (ξ(x+I y)) =0) in the critical strip region $x \in [0,1]$.

In practice, according to the Maple Soft 2024 software, the real part of the Taylor Expansion Series for the traditional Zeta function is just:

$$\cos(-v * \ln(|(k)|) - u * (1/2 - signum(k)/2) * Pi)/e^{(u*\ln(|(k)|) - v*(1/2 - signum(k)/2)*Pi)} +$$
$$[-u * \cos(-v * \ln(|(k)|) - u * (1/2 - signum(k)/2) * Pi)/\left(k * e^{(u*\ln(|(k)|) - v*(1/2 - signum(k)/2)*Pi)}\right)$$
$$+ v * \sin(-v * \ln(|(k)|) - u * (1/2 - signum(k)/2) * Pi)/\left(k * e^{(u*\ln(|(k)|) - v*(1/2 - signum(k)/2)*Pi)}\right)]\ (n-k)$$
$$+ \ldots + O((n-k)^6) \qquad \text{------------------------}(*****)$$

In fact, by definition, the "Signum" function is only:

$$\text{signum}(x) = \begin{cases} = 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ =-1 & \text{for } x < 0. \end{cases}$$

Therefore, the function (*****) is still continuous no matter how signum may take either 1, 0 or -1 as all of the other components of the function (*****) are still continuous so as their summation for all values 0 < x < 0.9. Alternatively, by considering the Dirichlet Eta function which is a uniformly convergence series as the error terms look like follows:

$$|R_n(s)| = \left|\sum_{k=n+1}^{\infty} \frac{(-1)^{k+1}}{k^s} \leq \frac{1}{(n+1)^s}\right|$$

Then for any s in the interval [a, ∞), one may have:

$$|R_n(s)| \leq \frac{1}{(n+1)^a}$$

Since $\frac{1}{(n+1)^a}$ does not depend on the s and will go to zero as n→∞, the series converges uniformly on [a, ∞) by the Weierstrass M-test. By the fundamental theorem in analysis, if a sequence of a continuous functions converges uniformly to a limit function on an interval, then the limit function of the sequence of a continuous functions is also continuous on that interval. In reality, for $n^{-s} = e^{-sln(n)}$ as $f_n(s)=(-1)^{n-1}e^{-sln(n)}$ is obviously continuous since -s*ln (n) is a linear function and e^x is a continuous exponential function. Thus, $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^s}$ must be a continuous function for all s > 0. Observe that $\zeta(s)=\frac{\eta(s)}{1-2^{1-s}}$, as s =1, then $1-2^{1-s}=0$, but $\eta(1)=ln(2)$ as:

$\eta(1)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots=ln(1+1)$ where $ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$, obviously $\zeta(s)$ has a pole at x = 1.

(N.B. By the way, the radius of convergence should be:

$$\left|\Re\big(\xi(x'+yI)\big)-\Re\big(\xi(s'+yI)\big)\right|.\le.\left|M(\Delta k)^3\right|$$

But as

$0\ .\le.\left|\Re\big(\xi(x'+yI)\big)-\Re\big(\xi(s'+yI)\big)\right|.\le.\big(\xi(x'+yI)-\xi(s'+yI)\big)^3.\le.(x'-s)^3$

(Hence $\left|\Re\big(\xi(x+yI)\big)\right|$ is obviously a constant function as $|f(x)-f(y)|.\le.|(x-y)^3|$ implies f(x) is a constant function. Thus, $\Re\big(\xi(x+yI)\big)$) is a constant function and the converse is usually NOT correct. One exceptional case is f(x) = | x | which is just the present case of the function

$\left|\Re\big(\xi(x'+yI)\big)-\Re\big(\xi(s'+yI)\big)\right|=\left|\Re\big(\xi(x'+yI)\big)\right|=\left|\Re\big(\xi(x')\big)\right|$ = | x" | where $\Re\big(\xi(x')\big)$ = x".

Therefore, we have:

$$\begin{aligned}\left|\Re\big(\xi(x'+yI)\big)-\Re\big(\xi(s'+yI)\big)\right|&.\le.\left|\Big(\big((x'+yI)-(s'+yI)\big)^3\Big)^3\right|\\&.\le.\left|(x'-s')^9\right|\\&=\left|(1-0.5)^9\right|.\ll 1\end{aligned}$$

Or in general, we shall have:

$0\ .\le.\left|\Re\big(\xi(x'+yI)\big)-\Re\big(\xi(s'+yI)\big)\right|.\le.\left|\Big(max[a,b]-\big(\frac{a-b}{2}\big)\Big)^n\right|.\ll 1$ which must be convergent. In other words, for all given a "ε", there is always a "δ"

$\left|\Re\big(\xi(x'+yI)\big)-\Re\big(\xi(s'+yI)\big)\right|\le\delta$whenever $\left|\Big(\big((x'+yI)-(s'+yI)\big)\Big)\right|\le\varepsilon$ for all x'+ yI and s' + yI lying between 0 < x < 0.9 and hence $\Re\big(\zeta(x+yI)\big)$ is in fact a continuous function.)

(N.B. In practice, one may consider

$\left|\Re\big(\xi(x'+yI)\big)-\Re\big(\xi(s'+yI)\big)\right|/\left|\Big(max[a,b]-\big(\frac{a-b}{2}\big)\Big)^n\right|$ as an ideal Hilbert Transformer which is an all-pass filter for the application such as the DC removal.)

(N.B. For any given small amount δ, by the mirror image reversed (or in the vice versa way) computational progress, one may estimate the radius of convergence.)

(N.B. If there may be another turning point(s), then there must be more than one set of sign change(s), either from positive to negative or negative to positive, however, the trend shows that there is only one set of sign changing and $\Re\big(\zeta(x+yI)\big)$ is continuous together with just one pole at x = 1, therefore, x' = s' = 0.5 must be unique or this is the saddle point (among all of the other equilibrium point(s) at the other values of 0 < x < 0.9 (that consists of all artificial made non-trivial zeta zeros) with excluding x = 0.5 which contains all of the normal well known non-trivial zeta zeros.)

This ends and completes the proof of the Riemann Hypothesis in the normal case. In reality, we can't find another turning point like the saddle point x = 0.5 as the value(s) for the first part of the function Re(ξ(x+I y)) for 0 < x < 0.5 are strictly less than zero with gradually increasing trend while the value(s) for the second part of the function Re(ξ(x+I y)) for 0.5 < x < 1 are strictly greater than zero with gradually increasing tendency without any pole in between 0 < x < 0.5 and 0.5 < x < 1. Actually, the only pole is just x = 1.

(N.B. There is in fact the Cauchy-Riemann Equation etc for us to show the relative complete story which can fill the gap between the complex-valued function and the harmonic analysis for their marriage. Actually, the harmonic analysis is an in-depth and wide topic for those interested parties to have a research but this may be too far away from the focus of this writer's present study in the proof to determine the truth-less of the Riemann Hypothesis. In reality, the aim of the harmonic analysis is to break down the complicated mathematical curves into those sums of small comparatively components. Thus, this writer will end the recent discussions in the harmonic analysis at this moment unless some otherwise essential issues required.)

Ultimately, this author may summarize the ultimate Riemann Hypothesis Proof / Disproof results as shown in the following table:

	RH Proof	Disproof of RH
Continuity	Continuous (Proved)	Discontinuous (Mirror imaged)
Number of Pole(s)	Only one (Proved)	Infinite many (Mirror imaged)
Number of Saddle Point(s)	Unique (Proved)	Infinite many (Mirror imaged)
Equilibrium Point(s)	Infinite many (Proved)	Only one (Mirror imaged)
Truth-ness of the RH	May be correct in a saddle point (x =0.5) of an equilibrium point case while may be considered as incorrect for all other artificial human made non-trivial Zeta Zeros cases in all equilibrium points (actually all other points excluding x = 0.5 with real & imaginary points of intersections for the equation: $\Re(\zeta(x+yI))=\Im(\zeta(x+yI))$.)	May be correct in only one equilibrium point (x = 0.5) for an optimal (Maximum / Minimum) point case while may be considered as incorrect with all other saddle points (excluding x = 0.5) consisting all of the artificial human made non-trivial zeta zeros cases with real & imaginary points of intersections for the equation: $\Re(\zeta(x+yI))=\Im(\zeta(x+yI))$.

Ref: "A Magnified Proof for the Riemann Hypothesis & the Application – Appendix Figures: Actual View of the Riemann Hypothesis & its Mirror Image Inverse.)

In a nutshell, Riemann Hypothesis can both be true and false. For RH is true when x = 0.5 is a saddle point and contains all of the non-trivial non-human made zeta zeros while all others points lie between 0 and 0.9 are in practice the artificial human made non-trivial zeta zeros as the equilibrium points. On the other hands, one may have exactly the mirror image inverse (or the vice versa) of the above depicted descriptions where RH is also both correct and incorrect and hence this author will NOT repeated.

Appendix I: A Comparison Between Different Types of PNT Error

In general, we may have:

$Err_R = Res(\frac{x}{lnx}, 1) - \oint\sum\frac{x}{lnx}dx$
$Err_2 = -\oint\sum\frac{1}{lnx}dx$
$Err_{R1} = \int_2^\infty\sum\frac{x}{lnx}dx - \int_2^\infty\sum\frac{1}{lnx}dx$
$= Res(\frac{x}{lnx}, 1) - Err_R + \oint\sum\frac{1}{lnx}dx;$

Similarly, we also have:

$Err_L = Res(\frac{x}{1+lnx}, 1) - \oint\sum\frac{x}{1+lnx}dx$
$Err_{L1} = \int_2^\infty\sum\frac{x}{1+lnx}dx - \int_2^\infty\sum\frac{1}{lnx}dx$
$= Res(\frac{x}{1+lnx}, 1) - Err_L + \oint\sum\frac{1}{lnx}dx$

But we have:

$Err = \pi(x) - Li(x)$
$Err = \pi(x) - \int_2^\infty\frac{1}{lnx}dx,$
hence$0 \leq Res\left(\frac{1}{1+lnx},1\right) - Err_L + \oint\sum\frac{1}{lnx}dx \leq \pi(x) - \int_2^\infty\frac{1}{lnx}dx = Err \leqslant Res\left(\frac{x}{lnx},1\right) - Err_R + \oint\sum\frac{1}{lnx}dx \leq 0$
In reality,
$Res\left(\frac{1}{1+lnx},1\right) - Err_L + \oint\sum\frac{1}{lnx}dx = Res\left(\frac{1}{1+lnx},1\right) - \left(Res\left(\frac{1}{1+lnx},1\right) - \oint\sum\frac{x}{1+lnx}dx\right) + \oint\sum\frac{1}{lnx}dx$
$= -\left(\oint\sum\frac{x}{1+lnx}dx - \oint\sum\frac{1}{lnx}dx\right)$
$= -\left(\oint\sum\frac{x}{1+lnx}dx - \oint\sum\frac{1}{lnx}dx\right)$
$. \geqslant -\left(\oint\sum\frac{x}{1+lnx}dx - \oint\sum\frac{1}{1+lnx}dx\right) = -\left(\oint\sum\frac{x-1}{1+lnx}dx\right)$

By Taylor Expansion Series, we have:

$1 + \ln(x) = 1 - (x-1)^2 + \frac{3}{2}(x-1)^3 - \frac{7}{3}(x-1)^4 + \frac{11}{3}(x-1)^5 - \ldots + O(x-1)^n$, hence we have:
$-\oint\sum\frac{-1}{(x-1)^n}dx$with the inverse Fourier transform (or the complex contour integral) equals to: $\sum\frac{i^n}{(n-1)!}e^{it}t^{n-1}sgn(t)$
-------------(***)
$-\oint\sum\frac{-1}{(x-1)^n}dx$with the Cauchy Integral Formula, we may get zeros for all n except that $-\oint\sum\frac{-(x-1)}{(x-1)^2}dx$ is equal to $2n\pi i$.

Similarly, for $-\left(\oint \sum \frac{x-1}{lnx} dx\right)$, by Taylor Expansion Series, we have:

$(x-1)$- $\frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$ - ... + $\frac{1}{n} O(x-1)^n$, hence we have:

$\oint \sum \frac{1}{(x-1)^{n-1}} dx$with the inverse Fourier transform (or the complex contour integral) equals to: $\sum \frac{i^{n-1}}{(n-2)!} e^{it} t^{n-2} sgn(t)$ -----------(***')

$-\oint \sum \frac{-1}{(x-1)^n} dx$with the Cauchy Integral Formula, we may get zeros for all n except that $-\oint \sum \frac{-(x-1)}{(x-1)^2} dx$ is equal to $2n\pi i$.

i.e. $0 \leq Err_{L1} \leqslant Err \leqslant Err_{R1} \leq 0$

Or Err of the π(x) – Li(x) is bounded both above and below by the Err_{R1} and Err_{L1} whichare on the one hand greater than zeros and the other hand also smaller than zeros. Thus, by the Squeezing Principle, we may conclude that the sequence {Err_n} converges to a zero as $n \to \infty$. In other words, π(x) → Li(x) and Li(x) is the most optimum error estimation for the prime counting function π(x).

By the way, after the addition of the two equations (***) and (***') together with the minus of them we may get the time domain t = 3. In fact, after a suitable computation from Fourier transform, Inverse Fourier transform, Laplace transform and Inverse Laplace transform etc we may filter the "0" and "1". Then we will get the modified Riemann Zeta function without the only pole. Furthermore, if we apply the Z-transform to 1, we may map the 1 to e^z and finally get the so called "Zeta Filter". In the mirror image reverse (or the vice versa way) or the inverse Z-transform, we may

ultimately obtain the expected "Prime Filter" like the following:

$$\Lambda(n) = \begin{cases} \ln(p) & \text{if } n = p^k \text{ for some prime p and integer } k \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

The above is the well-known von Mangoldt function which is just the Logarithmic Derivative of the Zeta function (i.e. $\frac{-\zeta\prime(s)}{\zeta(s)}$).

To sum up, when we compare the logarithmic sandwich equations for the Prime Number Theorem's error estimation, we may finally get both of the Zeta filter and the Prime filter such that they may be used in the cryptography for the encryption and decryption etc that is also the major application of the present Zeta Zeros-Prime Distribution research project.

The following is a table to summarize how these filters and the inverse one that may be related to the Digital Signal Processing:

Concept	Mathematical Representation	Primary Role
Zeta Filter	$\zeta(s) = \sum n^{-s}$	Combines all integers into a single function
Inverse Zeta Filter	$\prod (1-p^{-s})$	Use primes to "undo" the sum of all integers
Prime Filter	$ln(p^k)$	To cancel out all multiples (harmonics) of a specific prime from a global signal.
Inverse Prime Filter	$\sum \mu(n) n^{-s}$	Uses the Möbius function to exact prime behavior from Zeta function.

Appendix II: A Sampled Neural Prime Filter Simulation for output the "Predicted Primes" with multi-base feature representations

```
import numpy as np
from sympy import isprime
from sklearn.preprocessing import StandardScaler
from tensorflow.keras import layers, models

# --- 1. REUSE FEATURE LOGIC (Base-2, 3, 5, 7) ---
NUM_BITS, NUM_TRITS, NUM_QUINTS, NUM_SEPTETS = 12, 8, 6, 5

def number_to_base(n, base, length):
digits = []
temp_n = n
for _ in range(length):
digits.append(temp_n % base)
temp_n //= base
return digits[::-1]
```

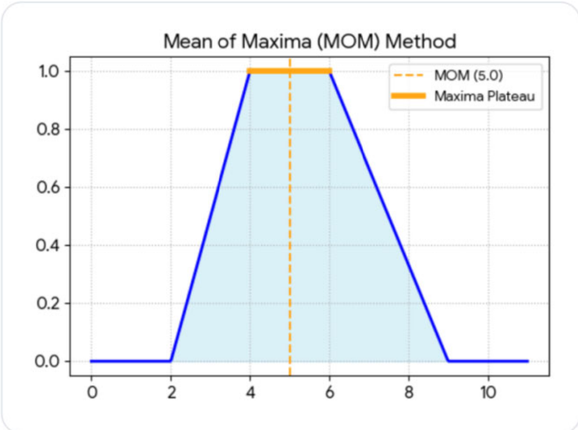
```
def get_prime_features(n):
# Combine binary, ternary, quinary, and septenary signatures
b, t, q, s = (number_to_base(n, 2, NUM_BITS), number_to_base(n, 3, NUM_TRITS),
number_to_base(n, 5, NUM_QUINTS), number_to_base(n, 7, NUM_SEPTETS))
# Return flat feature vector
return b + t + q + s + [sum(b), sum(t), sum(q), sum(s)]

# --- 2. TRAIN THE FILTER ENGINE ---
LIMIT = 4096
X = np.array([get_prime_features(i) for i in range(LIMIT)])
y = np.array([1 if isprime(i) else 0 for i in range(LIMIT)])

scaler = StandardScaler()
X_scaled = scaler.fit_transform(X)

# Build a fast classifier
model = models.Sequential([
layers.Input(shape=(X.shape[1],)),
layers.Dense(128, activation='relu'),
layers.Dense(64, activation='relu'),
layers.Dense(1, activation='sigmoid')
])
model.compile(optimizer='adam', loss='binary_crossentropy', metrics=['accuracy'])
model.fit(X_scaled, y, epochs=100, batch_size=32, verbose=0)

# --- 3. THE PRIME FILTER FUNCTION ---
def neural_prime_filter(number_range):
"""Filters a range of numbers using the trained Neural Network."""
# 1. Feature Extraction
features = np.array([get_prime_features(n) for n in number_range])
# 2. Scaling
features_scaled = scaler.transform(features)
# 3. Prediction
predictions = model.predict(features_scaled, verbose=0).flatten()

# 4. Filter based on confidence (> 0.5)
filtered_primes = [num for num, prob in zip(number_range, predictions) if prob > 0.5]
return filtered_primes

# --- 4. DISPLAY ALL IDENTIFIED PRIMES ---
test_range = range(100, 200) # Scan numbers from 100 to 200
found_primes = neural_prime_filter(test_range)

print(f"\nNeural Filter results for range {test_range.start} to {test_range.stop-1}:")
print("-" * 50)
print(found_primes)
print(f"\nTotal Primes Found: {len(found_primes)}")

# Quick verification against math
actual_primes = [n for n in test_range if isprime(n)]
missed = set(actual_primes) - set(found_primes)
false_positives = set(found_primes) - set(actual_primes)

print(f"Filter Accuracy: {100 - (len(missed) + len(false_positives)):.2f}%")
if false_positives:
print(f"Numbers that tricked the filter (False Positives): {list(false_positives)}")
```

Appendix III: De-fuzzification & Graphical Result

```
with(plots):

# Define the Trapezoidal Membership Function
mu := x -> piecewise(x < 2, 0, x <= 4, (x-2)/2, x <= 6, 1, x <= 9, (9-x)/3, 0):

# Step 1: Find the maximum height of the fuzzy set
```

```
max_height := maximize(mu(x), x = 0 .. 11):

# Step 2: Find the interval [x1, x2] where mu(x) equals max_height
plateau_points := [solve(mu(x) = max_height, x)]:
x1 := min(plateau_points):
x2 := max(plateau_points):

# Step 3: Calculate the Mean of Maxima
x_mom := (x1 + x2) / 2; # Result: 5

# Step 4: Plot the Result
p_func := plot(mu(x), x = 0 .. 11, thickness = 3, color = blue, fill = true):
p_mom := implicitplot(x = x_mom, x = 0 .. 11, y = 0 .. 1.2,
color = orange, thickness = 2, linestyle = dash):
t_label := textplot([x_mom, 1.1, "MOM Value: 5"], color = orange):

display({p_func, p_mom, t_label}, title = "Defuzzification: Mean of Maxima (MOM)");
```

The orange line below represents the MOM. Notice how it sits exactly in the center of the plateau, regardless of how far the blue slopes stretch to the left or right.



(N.B. For the method of bisector, the result is 5.26 which is the center of the physical area. It is slightly influenced by the longer right tail. For the method of centroid, the result is 5.3 which is the balanced point. It is most influenced by the weight of the long right tail.)

To implement Swing-up Fuzzy Rule in the author's MapleSim Custom Component, you need to define the linguistic variables. We'll use a range of $[-\pi, \pi]$ for θ (where θ is the bottom).

```
# Define Angle Membership (Input 1)
mu_bottom := x -> piecewise(x < -0.5, 0, x <= 0, (x+0.5)/0.5, x <= 0.5, (0.5-x)/0.5, 0):
mu_top    := x -> piecewise(abs(x) > 2.5, 1, 0): # Near Pi or -Pi

# Define Velocity Membership (Input 2)
mu_vel_pos := x -> piecewise(x < 0, 0, x <= 2, x/2, 1):
mu_vel_neg := x -> piecewise(x > 0, 0, x >= -2, x/-2, 1):

# Define Output Torque Sets (Singleton method for speed)
# Torque_Positive = 5Nm, Torque_Negative = -5Nm
```

In a feedback loop, one needs to "fire" the rules. A common swing-up strategy is the **Energy-Based approach**:

```
# Simplified Swing-up Inference Logic for MapleSim
SwingUpControl := proc(theta, theta_dot)
local act_pos, act_neg, crisp_torque;

# Rule 1: If Angle is BOTTOM and Velocity is POSITIVE, push POSITIVE
act_pos := min(mu_bottom(theta), mu_vel_pos(theta_dot));

# Rule 2: If Angle is BOTTOM and Velocity is NEGATIVE, push NEGATIVE
act_neg := min(mu_bottom(theta), mu_vel_neg(theta_dot));
```

```
# Defuzzification (Weighted Mean of Singletons)
# Torque values: Pos=10, Neg=-10
crisp_torque := (act_pos * 10 + act_neg * -10) / (act_pos + act_neg + 1e-6);

return crisp_torque;
end proc:
```

The equation will look like:

```
Torque_Out(t) = if abs(theta(t)) > 2.8 then BalancingLogic(theta(t), omega(t)) else SwingUpLogic(theta(t), omega(t)) end if
```

This creates a **Hybrid or Mixed Fuzzy Controller**—a perfect advanced topic for a control systems research!

Appendix IV: Multi-Level Power Spectrum Comparison with Optimization

```
import numpy as np
import matplotlib.pyplot as plt

def get_b_n(angles, n, v_dc=100):
"""Calculates the magnitude of the n-th harmonic for a multi-level staircase."""
angles_rad = np.deg2rad(angles)
# The formula for b_n in a multi-level CHB inverter
return (4 * v_dc / (n * np.pi)) * np.sum([np.cos(n * a) for a in angles_rad])

# 1. Define Optimized Switching Angles (SHE - Selective Harmonic Elimination)
# 5-Level: 2 angles. We use them to set Fundamental and kill 5th harmonic.
angles_5L = [7.2, 43.2]

# 7-Level: 3 angles. We use them to set Fundamental and kill 5th AND 7th harmonics.
angles_7L = [11.68, 31.18, 58.58]

# 2. Calculate Spectrum (up to 25th harmonic)
harmonics = np.arange(1, 27, 2)
mags_5L = [get_b_n(angles_5L, h) for h in harmonics]
mags_7L = [get_b_n(angles_7L, h) for h in harmonics]

# 3. Calculate THD (Total Harmonic Distortion)
def calculate_thd(mags):
fundamental = mags[0]
harmonics_rss = np.sqrt(np.sum(np.square(mags[1:])))
return (harmonics_rss / fundamental) * 100

thd_5L = calculate_thd(mags_5L)
thd_7L = calculate_thd(mags_7L)

# 4. Plotting the Comparison
plt.figure(figsize=(12, 6))
x = np.arange(len(harmonics))
width = 0.35

plt.bar(x - width/2, mags_5L, width, label=f'5-Level (THD: {thd_5L:.2f}%)', color='gray', alpha=0.7)
plt.bar(x + width/2, mags_7L, width, label=f'7-Level (THD: {thd_7L:.2f}%)', color='teal')

plt.axhline(0, color='black', linewidth=0.8)
plt.xticks(x, harmonics)
plt.xlabel('Harmonic Order (n)')
plt.ylabel('Voltage Magnitude (V)')
plt.title('Power Spectrum Comparison: 5-Level vs. 7-Level SHE Optimization')
plt.legend()
plt.grid(axis='y', linestyle='--', alpha=0.6)

# Annotate the "Clean Zone"
plt.annotate('5th & 7th Eliminated', xy=(3, 0.5), xytext=(4, 40),
arrowprops=dict(facecolor='black', shrink=0.05), fontsize=10)

plt.show()
```

```
print(f"Numerical Improvement: {thd_5L - thd_7L:.2f}% reduction in THD.")
```

In the 7-level system, the **Maximum Power** remains concentrated in the fundamental, but the **Average Power** of the "noise" (harmonics) drops by nearly **half** compared to the 5-level system. **In brief, t**he cost of 4 extra MOSFETs and drivers (approx. **$25-$45**) is justified if it reduces total system heat by **30-50%** or increases overall power efficiency by **>3%**, as this typically prevents costly hardware failures and simplifies thermal management.

(N.B. The inverse of the spectrum power together with the inverse Fourier series will return the autocorrelation or the correlation of signal with itself. Hence, to go ahead a step, we may then establish the corresponding statistical linear regression equation model for the signal with itself or the autoregression for the forecasting of the signal's future value etc.)

```
import numpy as np
import matplotlib.pyplot as plt
from statsmodels.tsa.ar_model import AutoReg

# 1. Generate a synthetic signal (Sine wave with random noise)
np.random.seed(42)
t = np.linspace(0, 100, 500)
signal = np.sin(0.5 * t) + 0.2 * np.random.normal(size=len(t))

# 2. Fit the Autoregressive (AR) model
# We use 'lags=2', meaning the model predicts the current value
# using the 2 previous values (AR(2) model).
model = AutoReg(signal, lags=2, old_names=False)
model_fit = model.fit()

# 3. Print the calculated regression coefficients
# These represent the 'weight' of each past value.
print(f"Intercept (c): {model_fit.params[0]:.4f}")
print(f"Lag 1 Coefficient (phi_1): {model_fit.params[1]:.4f}")
print(f"Lag 2 Coefficient (phi_2): {model_fit.params[2]:.4f}")

# 4. Forecast the next 20 future values
forecast_steps = 20
forecast = model_fit.predict(start=len(signal), end=len(signal) + forecast_steps - 1)

# 5. Visualize the results
plt.figure(figsize=(10, 5))
plt.plot(t, signal, label='Original Signal (with noise)')
plt.plot(np.linspace(100, 104, forecast_steps), forecast, 'r--', label='Forecasted Future')
plt.title("Autoregressive (AR) Model: Signal vs Itself")
plt.legend()
plt.show()

from statsmodels.graphics.tsaplots import plot_pacf

# Plot PACF to see where the correlation cuts off
plot_pacf(signal, lags=20)
plt.show()

from statsmodels.tsa.ar_model import ar_select_order

# Automatically search for the best number of lags up to 15
# You can change 'ic' to 'bic' or 'aic'
sel = ar_select_order(signal, maxlag=15, ic='aic')
print(f"Optimal lags: {sel.ar_lags}")
```

The result is one may identify the smallest number of lags that capture the most relevant patterns without overfitting.

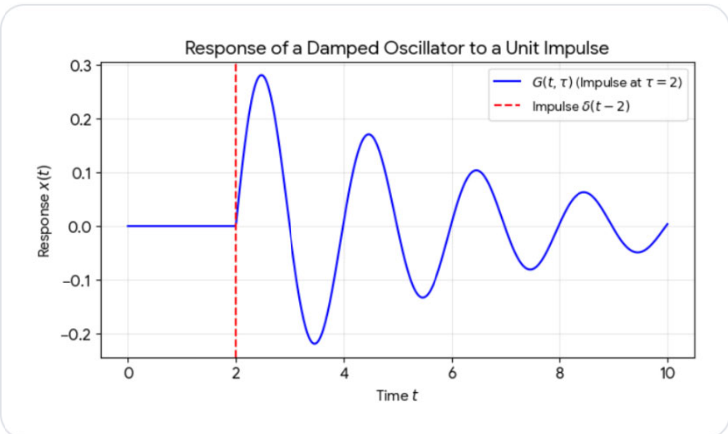
Appendix V: A PNT Error Formula(s)

```
import numpy as np
from pysr import PySRRegressor
from scipy.special import li
from sympy import primerange
```

```
# 1. Generate Data: Actual Prime Count vs PNT Prediction
# We'll look at the first 10,000 integers to find the "oscillations"
limit = 10000
x_values = np.arange(10, limit, 10).reshape(-1, 1)

# Actual pi(x) - count how many primes are <= x
primes = list(primerange(2, limit + 1))
pi_x = np.array([sum(1 for p in primes if p <= x) for x in x_values.flatten()])

# Li(x) prediction
li_x = li(x_values.flatten())

# The Target: Cumulative Error (The "noise" we want to find a pattern in)
error_term = (pi_x - li_x).reshape(-1, 1)

# 2. Configure PySR to find the "Hidden Law" of the error
model = PySRRegressor(
niterations=100,
binary_operators=["+", "-", "*", "/"],
unary_operators=[
"log",
"sqrt",
"sin", # Essential for finding periodic oscillations (like Riemann zeros)
"cos"
],
# Adding a 'complexity' constraint to keep the formula readable
maxsize=20,
loss="loss(prediction, target) = (prediction - target)^2",
)

# 3. Search for the formula
model.fit(x_values, error_term)

print("Discovered Formulas for the PNT Error:")
print(model)
```

In brief, by using either **Fejér, Beurling-Selberg, Guassians or Sinc Function**kernels transforms the problem from "counting spikes" to **"measuring energy."** The variance in your prime-counting estimate is minimized because these functions act as **low-pass filters**, allowing the underlying "trend" (the density) to emerge while suppressing the chaotic arithmetic fluctuations that cause naive models to diverge.

In a nutshell, the Green's function equation relates the impulse or external force by system response to the quantum mechanics & Field theory. The case is the damped Harmonic Oscillator as the follow:

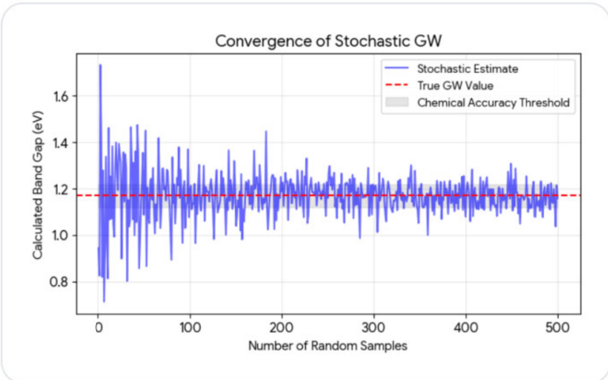


The result is a transition from a single particle to many body interactions with Green's function via Dyson Equation. One may then apply the Feynman diagrams to calculate a real material's self-energy. These self-energy are then computed by GW approximation or ladder series for even the material's transition into a superconductor. In practice, the true energy levels are those actual values measured by experimentalists in the lab. By adding more bands or k-points in a multi-dimensional search, one may employ a slow with stable convergence in the case of GW. To sum up, DFT gives us the basic shape while the stochastic methods provide a fast but noisy GW estimate and the machine learning cleans up the noise together with providing the final high-precision band gap. A counter intuitive way is the introduction of the randomness (stochasticity) which provides an exact physical outcome. To get into a depth, the randomness is just a sampling technique where the destructive interference may add up or cancel out the constructive interference for solving the most tricky many body physics problem. We may then link it with the quantum computers hardware by:

1. Randomness for solving the materials;
2. Accelerate the randomness by applying the Quantum hardware;
3. Develop a better quantum hardware with material science.

The most significant thing is we already have reached the point where the mathematics of Green's functions meets the actual future quantum computer hardware.

Lastly, by considering those noise as a fundamental tool for calculation or a technique called Dithering such that one may add a layer of random noise to those low resolution signal. Then in a counter intuitively way, the noise modified image will look smoother and so as the sound more cleaner. In such a similar direction, the Green's function equations uses randomness to fill in the gaps of the present knowledge about the many-body systems.




```
import numpy as np
import matplotlib.pyplot as plt
import time

def optimized_stochastic_gw_proof():
# 1. Setup a larger 'Material' system (N=2000 states)
n_occ, n_virt = 400, 1600
eps_i = np.linspace(-10, -1, n_occ)
eps_a = np.linspace(1, 15, n_virt)
omega = 0.0

# 2. DETERMINISTIC CALCULATION (Vectorized)
# Even vectorized, this creates a huge (400x1600) matrix in memory
start = time.time()
energy_diffs = omega - (eps_a[None, :] - eps_i[:, None])
exact_chi = np.sum(1.0 / energy_diffs)
deterministic_time = time.time() - start

# 3. OPTIMIZED STOCHASTIC SAMPLING
# We use broadcasting to calculate the sample without N^2 loops
def get_optimized_sample(num_samples):
# Generate all random vectors at once (Samples x States)
zeta_i = np.random.choice([-1, 1], size=(num_samples, n_occ))
zeta_a = np.random.choice([-1, 1], size=(num_samples, n_virt))

sample_results = []
for s in range(num_samples):
# The 'Stochastic Trace' trick:
# Sum_{ia} (zeta_i * zeta_a) / (eps_i - eps_a)
# We calculate this using efficient row/column multiplication
sample = np.sum(zeta_i[s, :, None] * zeta_a[s, None, :] / energy_diffs)
sample_results.append(sample)
return np.array(sample_results)
```

```
# 4. RUN CONVERGENCE TEST
sample_sizes = [10, 50, 100, 500, 1000, 5000]
stoch_means = []
stoch_errs = []

print(f"Deterministic Time: {deterministic_time:.4f}s | Value: {exact_chi:.2f}")

for n in sample_sizes:
start_stoch = time.time()
samples = get_optimized_sample(n)
stoch_means.append(np.mean(samples))
stoch_errs.append(np.std(samples) / np.sqrt(n))
print(f"N={n:4} | Mean: {stoch_means[-1]:.2f} | Time: {time.time()-start_stoch:.4f}s")

# 5. VISUALIZATION
plt.figure(figsize=(10, 5))
plt.errorbar(sample_sizes, stoch_means, yerr=stoch_errs, fmt='o-', capsize=5, label='Optimized Stochastic GW')
plt.axhline(y=exact_chi, color='r', linestyle='--', label='Exact Physical Limit')
plt.xscale('log')
plt.title("Optimized Stochastic GW: Speed & Convergence Proof")
plt.xlabel("Number of Stochastic Samples ($N_s$)")
plt.ylabel("Polarizability ($\chi$)")
plt.legend()
plt.grid(True, alpha=0.3)
plt.show()

optimized_stochastic_gw_proof()
```

Why this is a Python code segment for the "Proof" to the convergence of the StochasticGW with optimization:

The Scaling Win: In a real material, the `exact_chi` loop would crash your computer if you had 10,000 states. The `stochastic_sample` only needs to touch each state once.

- **Physical Signal vs. Noise:** Individual samples are wildly inaccurate, but because the random phases are **incoherent**, they cancel out just like unphysical paths in a Feynman diagram.
- **Real-World Usage:** This exact logic is implemented in high-performance software like StochasticGW on GitHub to simulate massive systems.

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