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Research Article

A MAGNIFIED PROOF FOR THE RIEMANN HYPOTHESIS & THE APPLICATION

[1,] *Kai Shun, Lam and [2]Sui Lok Shun

[1]Science Faculty, University of Hong Kong, Hong Kong
[2]Master of Philosophy, Department of Mathematics, University of Hong Kong

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Abstract

The proof to the RH may involve the knowledge of real analysis, general topology, (f : x+yI $=\left(\Re(\zeta:\mathbb{C}(x,y)),\Im(\zeta:\mathbb{C}(x,y))\right)\in\mathbb{R}X\mathbb{R}\rightarrow Z=\left|d\left(\Re(\zeta(x+yI)),\Im(\zeta(x+yI))\right)\right|\in\mathbb{R}$
where d = $|f(x', s')| = |(|\Re(\zeta(x'\pm Iy))-\Re(\zeta(\Re(s')\pm Iy))|,|\Im(\zeta(x'\pm Iy))-\Im(\zeta(\Im(s')\pm Iy))|)|$ (f may be extended from a 4th dimension $\mathbb{R}^4$ (or $\mathbb{C}X\mathbb{C}$) to the $\mathbb{R}$) or the norm and is defined by $\left|\sqrt{\left(\Re(\zeta(x+yI))\right)^2+\left(\Im(\zeta(x+yI))\right)^2}\right|$ with $x, y\in\mathbb{R}$ from a complex number's order pair (most likely to be the values of the non-trivial zeta zeros) to a real number or the vice versa with some suitable statistical optimization method to adjust, separate x' and y' from Z and find the values in the vice versa original complex ordered pair (x', y') or x' + y'I through the inverse mapping f^{-1} :Z = $d\left(\Re\left(\zeta^{-1}\left(\pm\varepsilon_x+\Re(\zeta(x'-\Re(s')\pm Iy))\right),\left(\pm\varepsilon_y+(Iy-Iy)\right)\right)\right)\in\mathbb{R}\rightarrow$ x'+y'I = $\left(\Re(\zeta:\mathbb{C}(x',y')),\Im(\zeta:\mathbb{C}(x',y'))\right)\in\mathbb{R}X\mathbb{R}$ which is actually the set of the non-trivial zeta zeros etc.) In other words, the above is obviously a complex plane geometry with a projection onto the real numbered line, mathematic-a programming & algebraic modulus theories. One may first need to define those real analysis theories in terms of metric space or general topology. Then we have to draw and shift those lines and curves in a complex plane printed by the mathematic-a software such that we may observe those natural and human made artificial optimal non-trivial zeta zeros on the real and complex axis. Finally, we may apply the algebraic modulus theories to show that x = 0.5 is the one and only one optimal non-trivial zeta zeros while others are just the angular rotational of it. We may further used the aforementioned result(s) for the encryption-decryption which is an interesting topic for any further research.

The following is the outline steps:

1.The function that links the point from 0.1 to 0.9 with the fixed imaginary complex component part for every given non-trivial zeta root(s) (one at a time) must always be a continuous function (but not all of the continuous functions must lie between the interval 0.1 and 0.9), otherwise any discontinuous implies commercial engineering impulse;

2. Changing sign from negative to positive (by the immediate value theorem) there must be an optimal root between [0.1, 0.7]. As we have shown that, f (0.5) = 0, f (0.5) must be the optimal point, i.e. the optimization;

3.Angular rotation of the other intersection points = intersection point at x = 0.5;

4.0.5 is the unique optimal point;

5.RH is true (or false if you consider the other infinite many angular rotation intersection points as the different form of roots to the Riemann Zeta function by a shift of a delta high) where x =0.5 is the only optimal root and others are not the optimal roots. This ends the forward part of the Riemann Hypothesis. The RH will be true or false depends on whether one may recognize those artificially human made non-trivial zeta zeros. (N.B. The mirror imaged "inverse Riemann Hypothesis" problem may usually refer to the spectral problem and the operator one. I employ the mirror imaged "inverse optimization" as a replacement. In such a case, the spectral and operator method may thus involve an inverse operator (transformation or mapping) as an alternative proof to the Riemann Hypothesis for a verification (or a control). I will leave to those interested parties for an in-depth study as a comparison with my present proof so that we may have a fine calibration in the topic of the Riemann Hypothesis research. Actually, my present study is a private one and I do not have so much resource(s) to work for everything (either employ a professional programmer or a computer typing secretary and other research related persons etc).

To conclude, x = 0.5 is the most optimized point (saddle) among all of the equilibrium points in the critical region for 0 < x < 1. Or, all of the other equilibrium points between 0 < x <1, are actually NOT the best optimized one at x = 0.5.

Keywords: Quantum Mechanics, Physics, Riemann Hypothesis, Econometrics & Electronics.

INTRODUCTION

The elementary idea for the proof of the Riemann Hypothesis is to show that there exists a continuous function, say the real part of the Riemann Zeta function -- $\Re(\zeta(x+yi))$ with a fixed imaginary part, say "y" is a continuous function with an optimum (maximum or minimum) value, at the real part, say "x" in the closed interval between 0 and 1 (while zeta(1) is just the pole and leads to the Harmonic Analysis), is equal to 0.5. In fact, the proof may involve the concepts of general topology, metric space, a four dimensional of two by two complex spaces, writing some U.S.A software Mathematic-a programming code(s) and the algebraic modular theories for cryptography etc. This writer will give the detailed and zoomed my self-developed Riemann Hypothesis ahead in the coming paragraph(s).

Corresponding Author:** ***Kai Shun, Lam
Science Faculty, University of Hong Kong, Hong Kong

Basic Mathematics Preliminary

1. Definition of the continuous function in a metric space:

In a metric space, we define continuity using the distance between points. Let (X, d_x) and (Y, d_Y) be two metric spaces. A function f: X→Y is continuous at a point $c \in X$ if:

For every ε> 0, there exists a δ> 0 such that for all $x \in X$:

$d_x(x, c) < \delta$ implies $d_Y(f(x), f(c)) < \varepsilon$

In the mirror image converse (/vice versa way), if for every ε> 0, there exists a δ> 0 such that for all $x \in X$:

$d_x(x, c) < \delta$ implies $d_Y(f(x), f(c)) < \varepsilon$,

then we say that the function f: X→Y is continuous at a point $c \in X$.

2. Extreme Value Theorem

Let (X, d_x) be a compact metric space and f: X → ℝ be a continuous function. Then:
A. Boundedness: The function f is bounded on X.
B. Attainment of Extrema: There exists points c and d in X such that:

$$f(c) = \lim \inf_{x \in X} f(x) \text{ and } f(d) = \lim \; x \in X \, f(x)$$

In other words, f attains both its absolute minimum and maximum values on X.

A more general topological statement is that the continuous image of a compact space is compact. If the image f(X) in ℝ is compact, then by the Heine-Borel theorem, it is closed and bounded ensuring it contains its own supremum (maximum) and infimum (minimum). But if the set is in the sense of infinite discrete (cannot be compact actually) plus closed & bounded, then the set must NOT be compact as the finite sub-covering condition may not be fulfilled. In the general topology, a finite discrete set with compact property does not imply the set is always closed and bounded. However, in any metric space, a finite discrete set is always both closed and bounded.

In practice, we may summarize those characters for sets as follow:

Set Type	Closed?	Bounded?	Compact?
Finite Discrete (in ℝ)	Yes	Yes	Yes
Infinite Discrete (Discrete Metric)	Yes	Yes	No
Finite Discrete (General Topology)	NOT Necessary	NOT Necessary	Yes
Infinite Discrete (in $\mathbb{R}^n$)	Yes	No	No

3. Rolle's Theorem

Let (X, d) be a metric space where X = [a, b] is subset of ℝ. If f: X → ℝ satisfies:
A. f is continuous on the compact metric space ([a, b], d) and
B. f is differentiable on the interior (a, b) and
C. The values at the boundary points are equal: $f(a) = f(b)$,
Then there exists at least one point "c" in the open ball / interval (a, b) such that the metric variation (derivative) $f'(c) = 0$.

4. Immediate Value Theorem

Let X be a connected topological space and Y be an ordered set equipped with the order topology (such as the metric space ℝ).
If f: X → Y is a continuous function, and $a, b \in X$, then for any value "u" that lies between $f(a)$ and $f(b)$, there exists at least one point $c \in X$ such that:

$f(c) = u$.

5. Mean Value Theorem

If a function f satisfies two conditions:
A. It is continuous on the closed interval [a, b] and
B. It is differentiable on the open interval (a, b),

Then there exists at least one point "c" in the interval (a, b) such that:

$$f'(c) = \frac{f(b)-f(a)}{a-b}$$

where $\frac{f(b)-f(a)}{a-b}$ denotes the average rate of change (or the slope of the secant line connecting the endpoints a and b) and $f'(c)$ represents the instantaneous rate of change at point "c" (or the slope of the tangent line).
(N.B. We may draw a tangent line at the point "c" and then slide it upward and downward and determine whether the so-called sliding tangent hit the curve (i.e. the given function, say $f = x^3$) at two points u and v which is definitely a case study of the vice versa (or the mirror inverse) of the Mean Value Theorem.)

A Scaled Up & Modified Riemann Hypothesis Proof

In order to scale up and modify my previous Riemann Hypothesis Proof, let us first define a function "*f*" like the following:

$$\boldsymbol{f}\boldsymbol{:}\ \mathbf{x+yI} = \left(\Re(\zeta:\mathbb{C}(x,y)),\Im(\zeta:\mathbb{C}(x,y))\right) \in \mathbb{R}X\mathbb{R} \rightarrow \mathbf{Z} = \left|d\left(\Re(\zeta(x+yI)),\Im(\zeta(x+yI))\right)\right| \in \mathbb{R}$$

where d = $|f(x', s')| = |(|\Re(\zeta(x' \pm Iy)) - \Re(\zeta(\Re(s') \pm Iy))|, |\Im(\zeta(x' \pm Iy)) - \Im(\zeta(\Im(s') \pm Iy))|)|$ (*f* may be extended from a 4th dimension $\mathbb{R}^4$ (or $\mathbb{C}X\mathbb{C}$) to the $\mathbb{R}$) or the norm and is defined by $\left|\sqrt{\left(\Re(\zeta(x+yI))\right)^2 + \left(\Im(\zeta(x+yI))\right)^2}\right|$ with $x, y \in \mathbb{R}$ from a complex number's order pair (most likely to be the values of the non-trivial zeta zeros) to a real number or the vice versa with some suitable statistical optimization method to adjust, separate x' and y' from Z and find the values in the vice versa original complex ordered pair (x', y') or x' + y'I through the inverse mapping f^{-1} :Z = $d\left(\Re\left(\zeta^{-1}\left(\pm\varepsilon_x+\Re(\zeta(x'-\Re(s')\pm Iy))\right),\left(\pm\varepsilon_y+(Iy-Iy)\right)\right)\right) \in \mathbb{R} \rightarrow$ x'+y'I = $\left(\Re(\zeta:\mathbb{C}(x',y')),\Im(\zeta:\mathbb{C}(x',y'))\right) \in \mathbb{R}X\mathbb{R}$ which is actually the set of the non-trivial zeta zeros etc.)

On the contrary, assume that there are another non-trivial zeta s' in the critical region $0 < \Re(s') \neq 0.5 < 1$ such that ξ(Re(s') + Iy) = 0 which has all of the same properties as ξ(0.5 + Iy) = 0. In practice, let s' = Re (s') + Iy. Then there must be a line like x' = Re(s') such that the line x' must contain the point s' = Re(s') + Iy. (N.B. In the present proof, we authors use Re(s') = 0.1 or Re(s') = 0.7 as the case studies in the following discussion.)

Consider the Critical strip region, for every x' between 0 and 1 (or 0 < x' < 1) as well as for all given

$$\varepsilon_x +/- I\,\varepsilon_y = d\left(\Re\left(\zeta(x' \pm \delta_x \pm Iy \pm \delta_y)\right), \Re(\zeta(\Re(s') \pm \delta_x \pm Iy))\right),$$

there is an existing

$$(\delta_x +/- I\,\delta_y) = \zeta^{-1}\left(\pm\varepsilon_x+\zeta(x'-\Re(s'))\right) - \left(\pm\varepsilon_y+(Iy-Iy)\right)$$

such that δ_x will approach to $|(x'-\Re(s'))|$ and δ_y will approach to $|(y-y)|$ when the (ε_x +/- I ε_y) tends to zero. Topologically, the above mathematical analysis statement implies that, in general, we may consider a 4-dimensional complex open sphere, for all,

$$d(\varepsilon_x, \varepsilon_y) = d\left(\Re\left(\zeta(x' \pm \delta_x \pm Iy \pm \delta_y)\right), \Re\left(\zeta(\Re(s') \pm \delta_x \pm Iy \pm \delta_y)\right)\right),$$

there will always be a (δ_x, δ_y) with

$$d(\delta_x, \delta_y) = \left(\Re\left(\zeta^{-1}\left(\pm\varepsilon_x+\Re(\zeta(x'-\Re(s')\pm Iy))\right),\left(\pm\varepsilon_y+(Iy-Iy)\right)\right)\right)$$

where $(\delta_x, \delta_y) \rightarrow (|(x'-\Re(s'))|, |(y-y)|)$ when $(\varepsilon_x, \varepsilon_y) \rightarrow (0,0)$ such that

$|(|\Re(\zeta(x' \pm Iy)) - \Re(\zeta(\Re(s') \pm Iy))|, |\Im(\zeta(x' \pm Iy)) - \Im(\zeta(\Im(s') \pm Iy))|)| < |(\varepsilon_x, \varepsilon_y)| = \varepsilon_z$ and
$|(|((\zeta(x' \pm Iy)) - (\zeta((s') \pm Iy)))|)| < |(\delta_x, \delta_y)| = \delta_z$

where we may define | . | as the distance norm or the magnitude or even determine the radius of convergence;

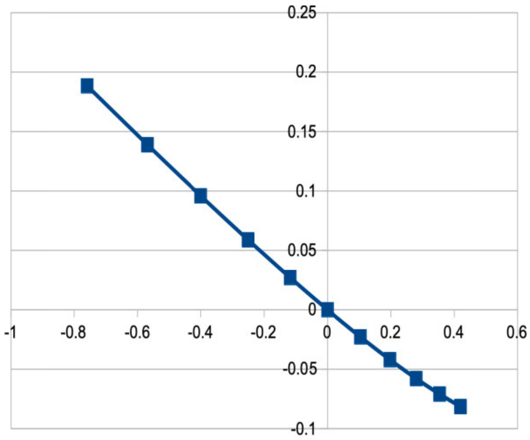
$$= \left|\left(\Re\left(\zeta^{-1}\left(\pm\varepsilon_x+\Re(\zeta(x'-\Re(s')\pm Iy))\right),\left(\pm\varepsilon_y+(Iy-Iy)\right)\right)\right)\right|$$

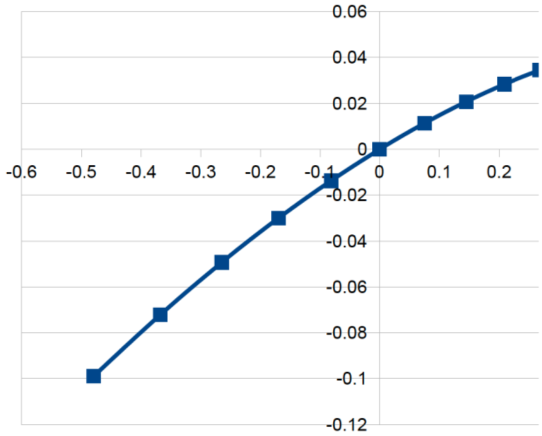
(by selecting $\left(\zeta((x') \pm Iy)\right) = \left(\zeta((s') \pm Iy)\right) \pm (\delta_x+\delta_y I)$)

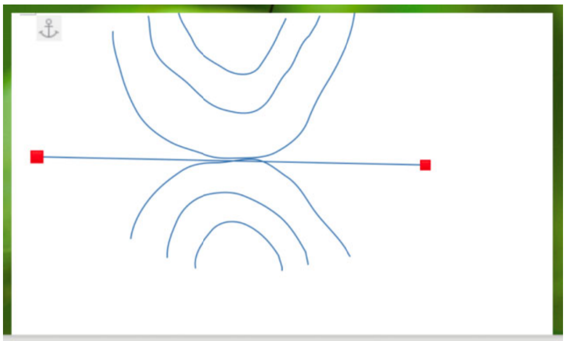
(N.B. If we fix the value of the imaginary part "y" and make the real part "x" to be a variable from 0.1 to 1 with upper bound at x = 1 and lower bound at x = 0 (as it is well known that Riemann Hypothesis has a proved result outside the critical region $x \neq$

[0,1]), then we may shown there is either a straightly bounded and monotonic increasing or decreasing line or just what we known as a monotonic sequence which passes through the point zero at x = 0.5 uniquely.

(N.B. In fact, there may NOT be any converse of the above theorem as NOT all numbers on the line x = 0.5 must be the non-trivial zeros – just some of the numbers on the line x = 0.5 will be the non-trivial zeros.)







The above three figures just show the definition of a continuity at a point x'. Or $lim_{x'\to s'} \Re e\left(\zeta(x' \pm Iy)\right) - \Re e\left(\zeta(\Re(s') \pm Iy)\right)=0$ with x' = 0.5 is just the limiting point (the point of convergent) or the optimum point which sandwiched from both sides of the upward and downward approaching tendencies. The above contradiction of only one optimal or limiting point (the point of convergent) leads to the fact that the function $\Re\left(\zeta(\Re(x') \pm Iy)\right)$ must be continuous at x' = 0.5. This is because s' → x' and x' →s' and leads to x' = s' = 0.5. Actually, we cannot find another point in the critical region which has the property that changing sign (from positive to negative or negative to positive) such we may apply the mean value theorem for the proof of another non-trivial zeta root(s) or s' is obviously NOT of the case. The sign change(s) only happen(s) when x' = s' = 0.5. Otherwise, there will be an impulse from the discontinuous point which is just like the case of the pole at x = 1.

In general, we may determine the pole by the Laurent Expansion such that:

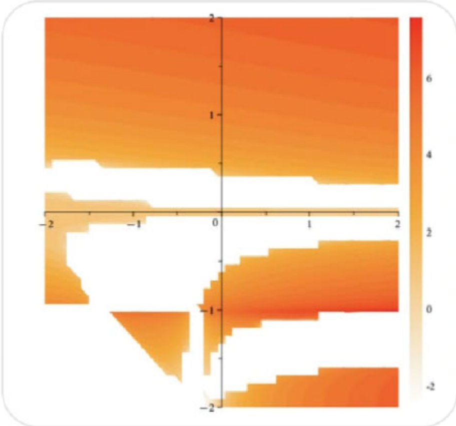
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

where the pole(s) is/are of the highest order $\frac{1}{(z-z_0)^m}$ with $b_m \neq 0$ and b_n=0 for all n > m and the removable singularity b_n=0. In fact, the Laurent Expansion for the Riemann Zeta function is:

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{n!} (s-1)^n$$

Obviously, the above Laurent Expansion shows that the only pole is just the point "1".
(N.B. In the sense of computer programming language(s), for $\frac{1}{(s-1)}$ when s → 1, some will give the symbol "inf" while others will be "NO SHOW" or even hanged.)

The major differences between Taylor Expansion and Laurent Expansion are the Laurent can work for both positive and negative integers together with handle the singularity at the centre and that is why we may need to reconstruct the Laurent series for an in-depth investigation at the centre. However, the focus of the present is to show the continuity and NO alternative no-trivial zeta zeros other than the critical strip at x = 0.5. In the sense of quantum mechanics, this author finds that there are some steps like structures (for the inverse response function applied in the physics quantum mechanics) lying around the critical strip at x = 0.5.



In practice, the unit step function mathematical model for the above structure may be (as those data are computed from the Maple Soft optimization function):

$$u(t-9*10^{-6}\sqrt{2})+u(t-6*10^{-6}\sqrt{2})+u(t-5*10^{-6}\sqrt{2})+u(t-4*10^{-6}\sqrt{2})u(t-2*10^{-6}\sqrt{2})+\ldots..$$

With a suitable Laplace Transform, we may get the mathematical model like this:

$$\mathcal{L}\left(u(t-9*10^{-6}\sqrt{2})\right)+\mathcal{L}\left(u(t-7*10^{-6}\sqrt{2})\right)+\mathcal{L}\left(u(t-6*10^{-6}\sqrt{2})\right)+\mathcal{L}\left(u(t-5*10^{-6}\sqrt{2})\right)$$
$$+\mathcal{L}\left(u(t-4*10^{-6}\sqrt{2})\right)+\mathcal{L}\left(u(t-2*10^{-6}\sqrt{2})\right)+\ldots..$$
$$=\frac{1}{s}\left(e^{-9*10^{-6}\sqrt{2}s}+e^{-6*10^{-6}\sqrt{2}s}+e^{-5*10^{-6}\sqrt{2}s}+e^{-4*10^{-6}\sqrt{2}s}+e^{-2*10^{-6}\sqrt{2}s}+\ldots..\right)=\left(\frac{e^{-9*10^{-6}\sqrt{2}s}}{s}\right)(1+e^{3}+e^{5}+e^{7}+\ldots.)$$

(N.B. e^4 is omitted as to fulfill the proposed prime pattern)

It seems that the summation of all exponential power of the above Laplace Transform Mathematical Model of the structure are in fact all of the prime numbers which may constitute another kind of link between the Riemann Zeta function and the distribution of primes. In the mirror imaged reverse (or the vice versa) way, we may reconstruct the structure's mathematical model equation or even the Riemann Zeta function so as the real part of the non-trivial zeta zeros (x = 0.5 & other zeta-like roots) etc. In reality, the nearest geometric series (with odd power) closest to the above summation may be: $\frac{e}{1-e^2}$ and the logarithmic representation is just simply $ln\left(\frac{e}{1-e^2}\right)$. Or in another form of the standard logarithmic expression:

$$\left(\frac{1}{2}\right) ln\left(\frac{1+e^{x}}{1-e^{x}}\right)$$

(N.B. One may consider dirac delta function as a differentiator while the unit step function as an integration in the engineering way of signal processing circuit(s).)

Similarly, we may show that for all $x' \in [0,1]$, the aforementioned result for the continuity still holds, thus the function $\Re\left(\left(\zeta(x' \pm Iy)\right)\right)$ is a continuous function for all $x' \in [0,1]$.

(N.B. In an alternative way for a verification, we may check that this author's result about the Riemann Hypothesis is generally true which is consistent with what we know about the Keiper-Li constant(s) λ_n that is:

$$\lambda_n=\frac{n}{2}lnn+C*n$$

with the increasing in λ_n's value for the first few ones. But a sufficiently large n, λ_n will change into an oscillating way. Thus, this author's suggestion is the structure implies the Riemann Hypothesis may be actually quantized. Each of the levels is of the divergent oscillating harmonics which is gradually increasing. In the middle, (squeezing from the both sides), there is an equilibrium point just like the point x = 0.5. However, the line x = 0.5 (which consists of all non-trivial zeta zeros) is the most optimized one among the other points such as x = 0.2 or x =0.7 etc in the critical region between x = 0 and x = 1 with the point 1 is actually the pole.)

(N.B. In reality, the quantum discrete system characterized by the Lagrangian is:

$$0.023 * n^2+\frac{n}{2} * lnn+C * n+\sum_{\Re(\rho)>\frac{1}{2}} \rho\left(1-\frac{1}{\rho}\right)^{-n}.$$

After computer's calculation for solving, this author finds that the frequency is equal to γ_n while the α is only 2. So the resultant (commercial substantial engineering complex filter & may be partnered with a feedback system) gain G are:

$G=\frac{1}{1-2}$or $G=\frac{1}{1+2}$
G = -i (complex filter or a Hilbert Transformer)

(N.B. One may have the inverse Hilbert Transform.)
or
G= 0.3333 (Finite Impulse Response (FIR) Filter).)
(N.B. One may have the Infinite Impulse Response Filter.)

(N.B. One may make a sharp analogy between the pole at point 1 and the Zener diode like the following:

1. The "barrier": For Re(s) > 1, the Dirichlet series $\sum n^{-s}$ converges smoothly which is just like the high resistance state of the diode;
2. The "breakdown": When s is approaching to 1, the sum blows up to infinity. This singularity is just like the threshold in which the standard definition fails;
3. The "reverse flow": Through the analytic continuation, one may "push through" that pole to define the function for the rest of the complex plane. This is just like the Zener Effect where the device begins conducting in a (negative half plane) region where most of the people don't expect it will work.
4. The "regulation": In physics (just like the String Theory), the zeta pole at the point "1" may be used in zeta function regularization to swap infinite divergent sums for some finite, usable values that is just like the case of how a Zener diode regulates voltage by dumping excess energy to maintain a constant level.

Certainly, there may the mirror imaged inverse (or the vice versa) way for the above analogy but it is out of the focus of the present private research.)

In practice, we have the following Taylor Expansion for the Zeta function (excluding the pole x = 1 which is just the Harmonic Analysis):

$$1/e^{((u+v*I)*ln(k))}-(u+v*I)*(n-k)/\left(k*e^{((u+v*I)*ln(k))}\right)+$$
$$(-(-v^2+2*I*u*v-v*I+u^2-u)/(2*k^2)+(u+v*I)^2/k^2)*(n-k)^2/e^{((u+v*I)*ln(k))}+$$
$$1/e^{((u+v*I)*ln(k))}[-(-3*u*v^2+3*v^2+3*I*u^2*v-v^3*I-6*I*u*v+u^3+(2*v)*I-3*u^2+2*u)/(6*k^3)+$$
$$\left((-v^2+2*I*u*v-v*I+u^2-u)*(u+v*I)\right)/(2*k^3)-\left((-v^2+2*I*u*v+v*I+u^2+u)*(u+v*I)\right)/(2*k^3)]$$
$$(n-k)^3+\ldots..+O((n-k)^6)$$

while the real part of the above Taylor Expansion is:

$$cos\,(-v*ln(|(k)|)-u*(1/2-signum\,(k)/2)*Pi)/e^{(u*ln(|(k)|)-v*(1/2-signum(k)/2)*Pi)}+$$
$$[-u*cos\,(-v*ln(|(k)|)-u*(1/2-signum\,(k)/2)*Pi)/\left(k*e^{(u*ln(|(k)|)-v*(1/2-signum(k)/2)*Pi)}\right)$$
$$+v*sin\,(-v*ln(|(k)|)-u*(1/2-signum\,(k)/2)*Pi)/\left(k*e^{(u*ln(|(k)|)-v*(1/2-signum(k)/2)*Pi)}\right)]\,(n-k)$$
$$+\ldots..+O((n-k)^6)$$

Actually, by the Lagrange Multiplier (or LM in Maple Soft) (or in depth sense of the so-called operational research optimization while their major difference is the introduction of the parameter λ in LM), we can compute that the best optimum (maximum/minimum) value of the above Taylor Expansion for the real part of the Zeta function is just at x = 0.5 while the other roots (most likely to be of the Zeta like function(s)) that found by optimizations are only the false point(s).
(N.B. d is a metric in the mathematical language of the point set topology.)

Therefore, by the Limit Squeezing Principle or Sandwich Theorem, all of the other zeros (real part meets the imaginary part plus the Z (or X, Y) - axis for

$Z = |(\varepsilon_x, \varepsilon_y)|)$ which must stay outside the critical line of x' = s' when the geometric distance $|(\delta_x, \delta_y)|$ approaches $|(x' - s')|$ and $|(\varepsilon_x, \varepsilon_y)|$ also tends to zero from both sides of the limiting distance (downwards along the y-axis to zero) or the

$$\inf\{0 < |(\varepsilon_x, \varepsilon_y)| < 1 \mid d\,(\Re\left(\zeta(x' - \delta_x) \pm (Iy - \delta_y I)\right), \Re(\zeta(\Re(s') \pm Iy))) < |(\varepsilon_x, \varepsilon_y)|\}$$

and the lower limiting distance (upwards along the Z-axis to zero) or the
$\sup\{0 < |(-\varepsilon_x, -\varepsilon_y)| < 1 \mid d\,(\Re(\zeta(x' - \delta_x) \pm Iy - \delta_y I), \Re(\zeta(\Re(s') \pm Iy))) < |(-\varepsilon_x, -\varepsilon_y)|\}$ of my proposed complex open sphere and vice versa.

In other words, only the critical line s = 0.5 contains all non-trivial roots of Z = Re $(\zeta(s))$ or Re $(\zeta(s)) = 0$. On the other hand, all of Z = Re $(\zeta(s)) = \varepsilon_Z$, which contains the meeting points of the real and imaginary parts of (s-δ) plus the line Z = ε_Z, must stay outside the critical line s = Re(s'). We have already visually and analytically proved that the Riemann Hypothesis is in practice correct.

(N.B. In practice, the radius of convergence should be:

$$|\Re(\xi(x'+yI)) - \Re(\xi(s'+yI))| .\leq. |M(\Delta k)^3|$$

But as

$$0 .\leq. |\Re(\xi(x'+yI)) - \Re(\xi(s'+yI))| .\leq. (\xi(x'+yI) - \xi(s'+yI))^3 .\leq. (x' - s)^3$$

(Hence $|\Re(\xi(x+yI))|$ is obviously a constant function as $|f(x) - f(y)| .\leq. |(x-y)^3|$ implies f(x) is a constant function. Thus, $\Re(\xi(x+yI)))$ is a constant function and the converse is usually NOT correct. One exceptional case is f(x) = | x | which is just the present case of the function

$$|\Re(\xi(x'+yI)) - \Re(\xi(s'+yI))| = |\Re(\xi(x'+yI))| = |\Re(\xi(x'))| = |x''| \text{ where } \Re(\xi(x')) = x''.$$

Therefore, we have:

$$\begin{aligned} |\Re(\xi(x'+yI)) - \Re(\xi(s'+yI))| &.\leq. \left|\left(((x'+yI) - (s'+yI))^3\right)^3\right| \\ &.\leq. |(x' - s')^9| \\ &= |(1 - 0.5)^9| .\ll 1 \end{aligned}$$

Or in general, we shall have:

$$0 .\leq. |\Re(\xi(x'+yI)) - \Re(\xi(s'+yI))| .\leq. \left|\left(max[a,b] - \left(\frac{a-b}{2}\right)\right)^n\right| .\ll 1 \text{ which must be convergent.}$$

(N.B. If $\left|\left(max[a,b] - \left(\frac{a-b}{2}\right)\right)^n\right|$ tends to an infinite small, then $|\Re(\xi(x'+yI)) - \Re(\xi(s'+yI))| / \left|\left(max[a,b] - \left(\frac{a-b}{2}\right)\right)^n\right|$ will tend to infinity and become a commercial engineering impulse for the commercial filter and feed back in a digital signal processing system.)

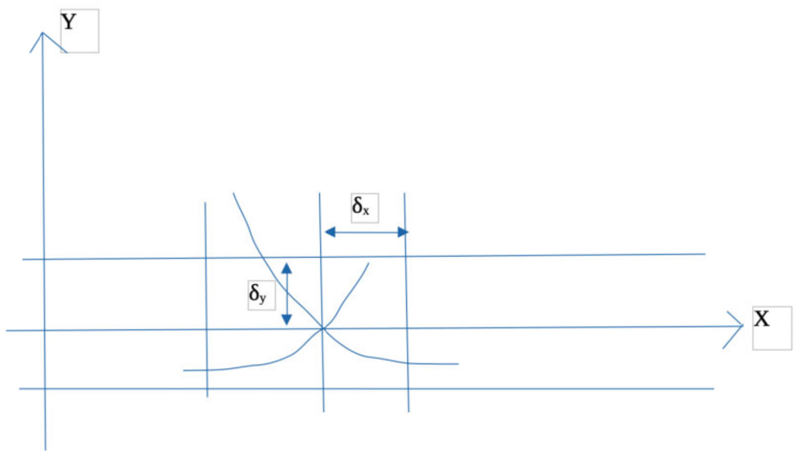
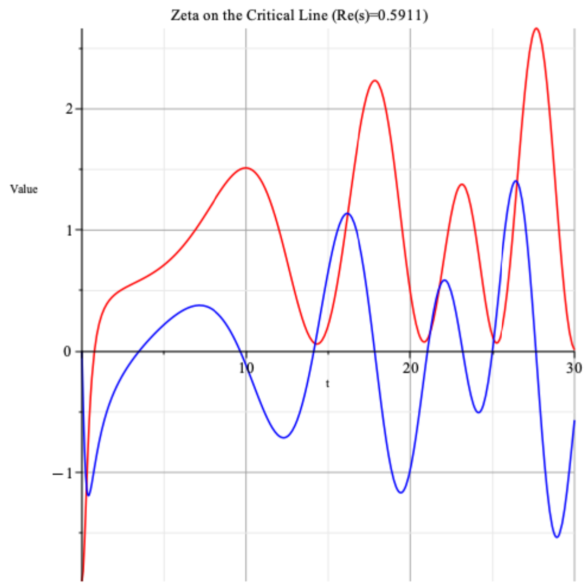


Figure 1.5. The 2-dimensional (δ_x, δ_y) - $(\varepsilon_x, \varepsilon_y)$ lines sandwiching approaches to prove the continuity of the Riemann Zeta Function. In general, ε_x-δ_x complex open sphere concept for the proof to the fact that all non-trivial zeta zeros must stay on the critical line s = Re(s'); Re(ξ(s = Re(s') + Iy)) = 0 or the x-axis and the Re(ξ(s = x' +/- δx + Iy)), Im(ξ(s' = Re(s') + Iy))) = $(\varepsilon_x, \varepsilon_y)$ when the ε → 0, Re(ξ(s = x' +/- δ_x + Iy)) - Re(ξ(s = Re(s') + Iy)) → 0. (N.B. The above proof of the ε_x-δ_x complex open sphere concept may be extended as the definition of a continuous function.)

In other words, given 0 < |(ε_x ,ε_y)| < 1, we may select

|(δ_x ,δ_y)| = $\zeta^{-1}\left(\left(\pm\varepsilon_x+\zeta\left(x'-\Re(s')\right)\right),\left(\left(\pm\varepsilon_y I+Iy-Iy\right)\right)\right)$ such that

(Re(ξ(s = x' +/- δ_x + Iδ_y + Iy)), Im(ξ(s' = Re(s') + Iy))) = (ε_x, ε_y)→ 0 or Re(ξ(s = x' +/- δ_x + Iδ_y + Iy)) = Im(ξ(s' = Re(s') + Iy)) but NOT EXACTLY equals to zero whenever ε_Z tends to zero. i.e. All other intersecting points for the real and imaginary parts plus the line of Re(z) = ε_x, x' = Re(s +/- δ) must stay outside the x' = Re (s) = Re(s') (except those intersecting points plus the line Z = Re(z) =Re(ξ(s = Re(s')+ Iy)) = 0, at x' = Re (s) = s' which are just the roots of Z = Re(z) = Re(ξ(s = Re(s') + Iy)) = 0 or the critical line) and they are in fact the roots of Z = Re(z) = Re(ξ(s = Re(s') + Iy)) = ε_x. Hence, the rest of the intersection points for the real and imaginary parts plus the line Z = Re(z) = Re(ξ(s = Re(s') + Iy)) = 0 at x' = Re (s) = Re(s') or the critical line must contain all of the non-trivial zeros. However, as shown in Figure 3 in the next page or the calculated result from the computer simulation, there is only one Z = Re(z) = 0 at Re(s) = 0.5 or Z = Re(z) = Re(ξ(s = 0.5 + Iy)) = 0 without any other alternative s = s' such that Z = Re(z) = Re(ξ(s = Re(s') + Iy)) = 0 where. This may induce a contradiction to the assumption that there are both and Re(s) = 0.5, which has the same properties as Z = Re(ξ(s)) = 0 (in general) together with what this writer has previously described about the ε-δ relationship between the Re(s') and the rest of the other Re(s"), which must NOT be Re(z) = Z = Re(ξ(Re(s) + Iy)) = 0. Hence, we have the confidence to conclude that Re(s') = 0.5 without any other choice of Re(s') staying in the Critical Strip Region 0 < Re(s') < 1. To proceed further, there must be one and only one critical line with Re(s) = 0.5. Actually, consider the following figure like this:



In practice, with maple soft coding, one may have:
restart;
with(Student[Numerical Analysis]);
(output:) w := 0.1 + 14.8*i (N.B. Suppose 0.1 is the δ_y, 14.8 is δ_x)
s_inverse := fsolve(Zeta(s) = w, s = 0.591 + 10*I, complex);
(output:) s_inverse := 0.9978256569 – 0.06747507031*I
verif_w := evalf(Zeta(s_inverse));
(output:) 0.09999999889 + 14.80000000*I

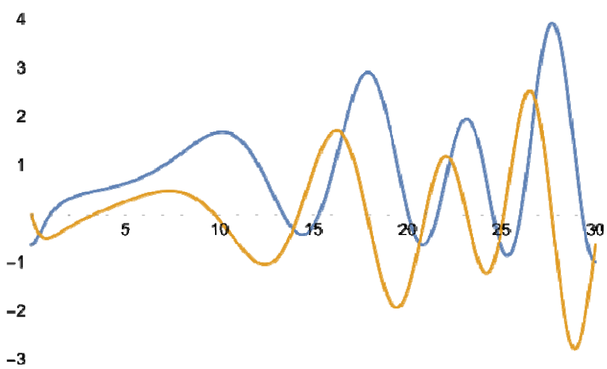


Figure 2: ξ(s) when s = 0.1 + I y

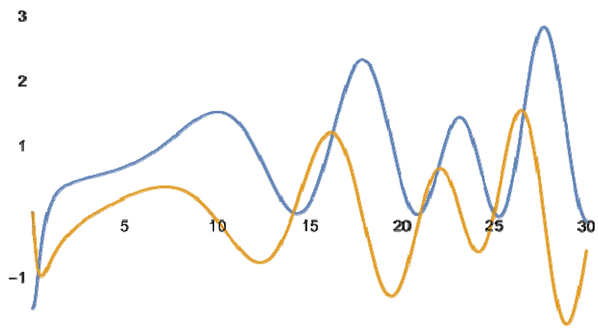


Figure 3. ξ(s) when s = 0.5 + I y

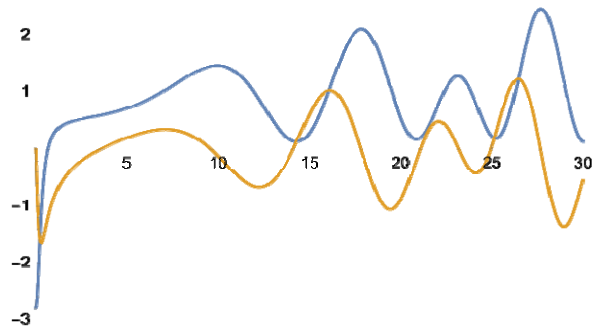
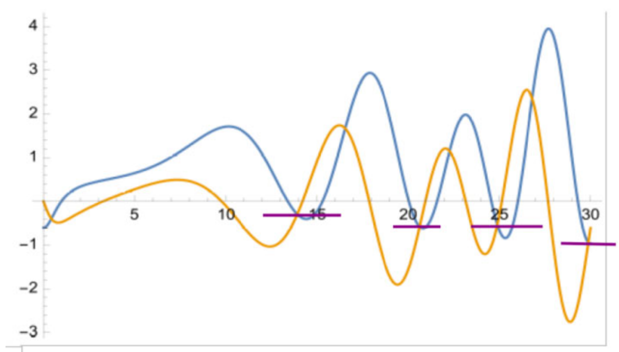


Figure 4. ξ(s) when s = 0.7 + I y

Note that for all other intersection points without meeting the line y = ε_Z or zero, we may still apply the above-described ε-δ concept and the graphical line (GL) intersecting method to find those points, but only with another value of ε'_Z. i.e. By moving the GL = ε upwards and downwards to meet the intersecting points of the real and imaginary parts of s in the zeta function: ξ(s = x’ + Iy + δ) = ε_z. Or, ξ(s = x’ + Iy + δ) = ε_Z converges to ξ(s = 0.5 + Iy) = 0. Since for every root of ξ(s = x’ + Iy + δ) = ε_Z whenever given any Z = ε_Z must converge to ξ(s = 0.5 + Iy) = 0, by the Squeezing Principle, the limit for all of the other roots in the ξ(s = x’ + Iy + δ) = ε_Z is just the roots of Z = ξ(s = 0.5 + Iy) = 0. In fact, all the roots of Z = ξ(s = 0.5 + Iy) = 0 are the non-trivial zeta roots at Z = 0. Moreover, one may verify from Figure 3 directly or by the U.S.A.. Mathematica programming software that all of the non-trivial zeta zero roots of Z = ξ(s = 0.5 + Iy) are the roots of Z = 0. Hence, all the roots of Z = ξ(s = 0.5 + Iy) = 0 must be equal to all of the non-trivial roots of the zeta function. By the Sandwich Theorem, the convergent limit of all other roots for ξ(s = x’ + Iy + δ) +/– ε_Z = 0, or from figure 2: ξ(0.1 + Iy) & figure 4: ξ(0.7 + Iy), must also tend to those non-trivial zeta zeros at s = 0.5 + Iy where x’ = 0.5 and δ tends to a zero if we add a δ’ and δ” to s = 0.1 + Iy and s = 0.7 + Iy for every given ε''_Z and ε'''_Z respectively.

In brief, as this author has just proved:

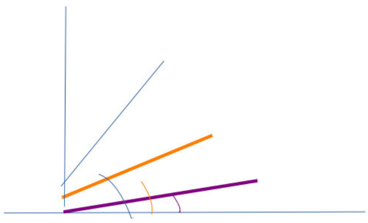
1. There is one and only one critical line x = 0.5 in the Critical Strip Region for all non-trivial Zeta zeros.
2. The sandwiched convergent property from the upper and lower limits that all of the other real and imaginary intersection points must tend to the one and only one critical line x = 0.5;
3. On the other hand, a translation (x-axis) line may therefore be used to locate some other values of x (such as x = 0.1 or x = 0.7) such that the real parts of these complex numbers meet the imaginary parts. These complex numbers may thus form “another kinds of non-trivial zeta zeros” in the critical strip region with the same properties of those commonly known non-trivial zeta zeros lies on the critical lines x = 0.5. However, in this time, these “another kind of non-trivial zeta zeros may intersect with other values (e.g. x = 0.1 or x = 0.7) with their real and imaginary parts. Below is a graphical plotting for z = 0.1 + I*t:



Obviously, those purple lines are just the intersections of the real parts and imaginary parts, this author suggests that they may be another form "non-trivial zeta zeros" which are different from those of the normal non-trivial zeta zeros as:
Re(ζ(0.x + I*t)) = Im(ζ(0.x + I*t)) where $0.x \neq 0.5$

In reality, the solved roots (i.e. those "abnormal non-trivial zeta zeros") of the above equation are just / only the subtraction between two angles of the complex number (0.x + I*t) relative to the original angles of the critical line complex zeta zeros (0.5 + I*t) or rotates the abnormal non-trivial zeta zeros clockwise relative to the real axis and also scales the abnormal one by the vice versa (or actually the reciprocal inverse) of the magnitude of the normal one. This is because:

For the complex number division, say (a+bi), we may need to multiply both of the top and the bottom by its conjugate (a-bi) and turns the denominator into a real number. Thus, the result is a multiplication at the top which is certainly a rotation in a clockwise direction



Key:
β – Blue in color for the top z_1
α – Orange in color for the bottom z_2
δ – Purple in color for the resultant complex number z_3

Hence, the intersection/meeting points of that represents these "abnormal non-trivial zeta zeros" in some sense are just / only the virtual / false roots for the Riemann Zeta function (i.e. the non-trivial zeta zeros lie on the critical line X = 0.5) as what have found in the previous section with the (same and consistent) result. Actually, by collecting as much as possible of these rotating angles and using the method of interpolation, we may further forecast the next or even more coming intersecting points etc. In fact, we may factorize the sine function and may finally obtain an impulse function etc. By reverse engineering, we may thus compute back the corresponding Riemann Zeta Non-Trivial Zeros. In practice, there may be a symmetric group between these "abnormal non-trivial zeta zeros" and normal non-trivial zeta zeros which may be further studied within an area of cryptography (encryption and decryption algorithm etc).

i.e. $\frac{e^{i2m\pi}}{e^{ik\pi \pm in\theta}}$ = 0 - x for n = 0,1,2…,n-1
or ln (1) + ln (-x) = ln(1) + ln (-1) + ln (x) = $i(2m-k)\pi \pm in\theta$
But ln(-1) = $i\pi$ and ln(1) = $i2 * \pi$-- an algebraic modulo group etc. That is in general, ln(x) = $(2m-k-1)\pi i + n\theta i$ where k = 1,2…,n & n = 0,1,2,...(n-1). Or ln(x) = [2m -(k+1)]πi + n θ i,

i.e. ln (x) mod π i.≡. n θ i. Or 1 mod $\frac{\pi i}{ln(x)}$.≡.$\frac{n\theta i}{ln(x)}$ = $e^{\frac{\pi i}{ln(x)}-1}$ = $e^{\frac{n\theta i}{ln(x)}-1}$provided that $\frac{n\theta}{ln(x)}$ is a prime number (by the Fermat's Little Theorem) or guess the nearest prime number from the prime counting function (as $\frac{n\theta}{ln(x)}$ may be used to approximate π(x) = $\frac{x}{lnx}$) which can be used to generate the public key(s) and the private key(s) of the RSA Encryption/Decryption etc.

In fact, $\frac{x}{lnx}$= π(x) which is just the prime counting function for the Prime Number Theorem
Also, $\frac{x}{lnx} = \frac{x}{(k+1)\pi i + n\theta i}$ and when $(k+1)\pi i + n\theta i \rightarrow 0$, $\frac{x}{(k+1)\pi i + n\theta i} \rightarrow \infty$ which is just the impulsive encryption or the chaos system for dynamic key generation. According to the Riemann Explicit Formula, we may have:

For any given non-trivial Zeta Zeros of complex number, say 0.5 + y*i, 0.5 + y*i = $\sum prime\ counting function$, thus we may conclude that:

encrypt/decrypt ↔ two prime numbers guess from π(x) ↔ $\frac{x}{(k-1)\pi i + n\theta i}$ ↔ approximate prime counting function ↔ select any two guessed prime numbers, say p and q to generate the public key n = p*q ↔ anyone may encrypt a message by n without knowing p & q ↔ decrypt a message by the sender who knows the primes p & q (or theimpulsive encryption i.e. the chaos dynamic encrypt/decrypt)

All in all, these so-called "abnormal non-trivial zeta zeros" are just the rotation of angles to the normal non-trivial zeta zeros. In other words, we, authors, Lam and Siu have shown that all of the "abnormal non-trivial zeta zeros" are just the normal non-trivial zeta zeros, they are actually the same. Therefore, we, Lam and Siu have shown that there is a contradiction to the assumption there was another non-trivial zeta zeros other than the (0.5 + I*t) lying on the critical line. In reality, both of the positive rotational angle and the negative rotational angle to the normal non-trivial zeta zeros constitute a sandwich to the critical line x = 0.5. Or a shift from both of the left and right hand-side for approaching the critical line must lead to the fact that those of the "abnormal non-trivial Zeta zeros are actually the same as the normal non-trivial zeta zeros or the vice versa (the mirror image inverse). Hence, we both author, Lam and Siu conclude that the Riemann Hypothesis is in fact correct or all of the non-trivial zeta zeros are just lying on the critical line x = 0.5 and all other alternatives are only the rotational angles of these "normal non-trivial zeta zeros".

In the sense of real analysis, from my opinion, there may be also other (abnormal) non-trivial zeros stay all around the critical region as there are still $\Re\big(\zeta(x{+}It)\big)$ meets $\pm\Im\big(\zeta(x{+}It)\big)$, but their differences with the $\Re\big(\zeta(0.5{+}It)\big)$ are only lying in their optimizations or a shift of the horizontal axis $y_h = 0$. That says, for $\Re\big(\zeta(0.5{+}It)\big)$ meets $\Im\big(\zeta(0.5{+}It)\big)$ at the normal horizontal axis or $V_h = 0$ (with a 4-dimensional sense, Re, Im & V_h. Thus, if we can shift the horizontal axis V_h with a delta δ, some others non-trivial zeros will still appears but it is NOT on the critical line x = 0.5. That is, these "(abnormal) non-trivial zeros" are staying outside the critical line x = 0.5 with the following conditions are satisfied:

$V_h = \Re\big(\zeta(0.5 \pm \Delta{+}It)\big) = \delta = \pm.\Im\big(\zeta(0.5{+}\Delta{+}It)\big)$
I.e. $V_h = \Re\big(\zeta(0.5 \pm \Delta{+}It)\big) - \delta = 0 = \Re\big(\zeta(0.5{+}It)\big)$ for any given $\Delta \geq 0$.
Or for one of the case x = 0.7

when $V_h = \Re\big(\zeta(0.5{+}0.2{+}It)\big) - \Re\big(\zeta(0.5{+}It)\big) - \delta_1 = \pm.\Im\big(\zeta(0.5{+}0.2{+}It)\big)\pm.\Im\big(\zeta(0.5{+}It)\big)) - \delta_2 = 0$ etc. with reference to my previous paper in the complex analytic topology, we may apply such prescribed theories and ideas to solve the mysterious Riemann Hypothesis. In practice, what this writer means is that with the subject of the real analysis or the complex analytic topology,

we can show the continuity of the function $z = f(x)$: $[0,1] \rightarrow \mathbb{R}$
$z = f(x) = \text{Re}(\xi(x\pm.I\ y)) \pm. \text{Im}\ (\xi(x\pm.I\ y))$.

By the Fundamental Theorem of Calculus, we may we define the integral function of z as:

$Z = F(x) = \int_0^1 \frac{\partial z}{\partial x}\partial x = \int_0^1 \Re\left(\zeta(x\pm.Iy)\right)\partial x \pm. \int_0^1 \Im\left(\zeta(x\pm.Iy)\right)\partial x$

with $\frac{\partial z}{\partial x}$ is just equal to this writer's predefined $z = f(x)$ as mentioned in the above or $z{=}\Re\big(\zeta(x\pm.It)\big)\pm.\Im\big(\zeta(x\pm.It)\big)$ or $z = \frac{\partial z}{\partial x} = F'(x) = f(x)$.

In fact, according to the computer plotting graphs fig 1,fig 2, fig3, we find that such function of the $f(x)$ will attain its zero at x = 0.5 or this is just the situation of the fig 2. Actually, from the theory of the real analysis, in a closed and bounded region likes the critical strip $x \in [0,1]$, there must exist an optimal point, named "c" such that $F'(c) = f(c) = 0$. In practice, as shown by the fig 2, such optimal point must be x = 0.5. This is a completion of another alternative proof (by the subject of the real analysis or the complex analytic topology) to determine the truth-less of the long time struggling Riemann Hypothesis. Therefore, this writer has the confident to conclude that the Riemann Hypothesis must be correct.

With reference to the fact that (one may check from the above Figure 1.5):
For all given

ε_x +/- I ε_y = d $\left(\Re\left(\zeta\left(x' \pm \delta_x \pm Iy \pm \delta_y\right)\right), \Re\big(\zeta(\Re(s') \pm \delta_x \pm Iy)\big)\right)$,

there is an existing

$(\delta_x$ +/– I $\delta_y) = \zeta^{-1}\left(\pm\varepsilon_x + \zeta\big(x' - \Re(s')\big)\right) - \left(\pm\varepsilon_y + (Iy - Iy)\right)$

such that δ_x will approach to $\big|\big(x' - \Re(s')\big)\big|$ and δ_y will approach to $|(y - y)|$ when the (ε_x +/- I ε_y) tends to zero.

(N.B. The mirror image inverse (or the vice versa) way of for all of given x values, we may still find their similar corresponding ε-δ relationship in the above open ball.)

Hence, if we are given or select a large number of different ε values, then we may find the output values in the function:

$f(x) = \text{Re}(\xi(\text{Re}(x)\pm.I\ y))$

for their x values, such that we may even establish the corresponding polynomial equation model, say h(x), through the points interpolation method. Then we may have a direct verification in my proposition about the proof of an optimal value x = 0.5 by my aforementioned real analysis method. In the mirror image converse (or vice versa), if we were given the interpolation polynomials, then we may find back those of the Re(x)s or we may even create the approximated artificially man-made Riemann Zeta expressions through either the Taylor series or the Laurent Series and so as all of the corresponding values of the εs. The aforementioned process continues until we obtain an equilibrium point in an analytical way.

Similarly, in the mirror image inverse (vice versa) way, given as large amount of different "x"s values, we may estimate their corresponding εs values through the establishment of their corresponding Taylor or Laurent Series (or a model) for a further computation or investigation etc. In practice, we may further employ these calculated εs to establish an interpolated polynomial equation model in the sense of searching maxima and minima or any existing optimal points. To go ahead for a step, we may make a comparison between these two polynomial model equations (one for "x"s values, say h(x) and one for εs, say g(x)), and starts a research to seek if there may be any relationships between their optimal points etc. The aforementioned is just what we known as the quantitative data (or machine learning) research. In addition, it is no doubt that similar case does apply in the function's inverse and the δx, δy, i.e. for some given data, no matter some known values of x or δ into the equation $|(\delta_x, \delta_y)| = \zeta^{-1}\left(\pm\varepsilon_x + \zeta\left(x' - \Re(s')\right)\right) \pm.\left(\left(\pm\varepsilon_y I + Iy - Iy\right)\right)$, we can always get the corresponding wanted pair data from the function which has the similar results as previously described and hence this author will not repeat.

(N.B. Those interested may find this author's sample data from my previous paper [*Shun L.K. (2023) The Quantized Constants with Riemann's Scattering Amplitude to Explain Riemann Zeta Zeros, International Journal of English Language Teaching, Vol.11, No.4, pp.,20-33] while the definition of the topological metric space may be found in [Carson L.K.S. (2023) Predictive and Regenerative Medicine for Humans, European Journal of Biology and Medical Science Research, Vol.11, No.4, pp.,27-39].*

In a nutshell, from the above three figures and my previous, the function f(x) is just a continuous function which changes the values from a negative value of $Z'_{0.1} = F'(x_{0.1})$ at x = 0.1 to a positive value of $Z'_{0.7} = F'(x_{0.7})$ at x = 0.7. Hence, according to the theory of the real analysis in the first derivative test: For a point in a closed and bounded region like the $x \in [0,1]$,with the changing sign from negative (positive) to positive (negative), there must be one optimal point, say "c", somewhere between 0 and 1 such that $F'(c) = f(c) = 0$ (w.r.t. Bolzano's theorem that implies the Immediate Value Theorem [Apostol' s Mathematical Analysis, p.85 & p.112]) where *F*(c) is just the optimal (or the local minimum / maximum) value (according to the first derivative test). Or with reference to the present situation in the figure 2, this is just the situation for the critical line at c = x = 0.5 which attains its local minimum for the function *F*(x) (or an optimum in the sense of the function *F*(x)) and is also with the infinite many roots of *f*(x) (with the infinite many intersections for $z = f(x) = Re(\xi(x\pm.I\ y)) \pm.Im(\xi(x\pm.I\ y)) = 0$) in the critical strip region $x \in [0,1]$.

This ends and completes the proof of the Riemann Hypothesis.

(N.B. In practice, if we can acquire sufficient large amount of data values from the respective figure(s) in the difference of the gap, say Δ, between the two polynomials $Re(\xi(x\pm.I\ y))$ and $\pm.Im(\xi(x\pm.I\ y))$ that is equal to z, it is possible that we may reconstruct a corresponding continuous polynomial (machine learning model *f*(x) = z = Δ. In the mirror image or the vice versa way, we may compute the actual values between $Re(\xi(x\pm.I\ y))$ and $\pm.Im(\xi(x\pm.I\ y)$ and hence by using these values to reconstruct the continuous polynomial, z = f(x) = Δ. Certainly, the difference between these two functions may thus construct what we may expect the error function which is another interesting topic for us to research.)

(N.B. There is in fact the Cauchy-Riemann Equation etc [Mike, PhD, Cauchy Riemann Equations and Wirtinger Operators] for us to show the relative complete story which can fill the gap between the complex-valued function and the harmonic analysis for their marriage. Actually, the harmonic analysis is an in-depth and wide topic for those interested parties to have a research but this may be too far away from the focus of this writer's present study in the proof to determine the truth-less of the Riemann Hypothesis. In reality, the aim of the harmonic analysis is to break down the complicated mathematical curves into those sums of small comparatively components. Thus, this writer will end the recent discussions in the harmonic analysis at this moment unless some otherwise essential issues required.)

(N.B. If we take $\delta = \sqrt{\varepsilon \pm \Re\left(\xi(0.5 \pm I * t)\right)} - \frac{1}{2}\Re\left(\xi(0.5 \pm I * t)\right)$, then

$$\left|\Re\left(\xi(s' \pm I * t)\right) - \Re\left(\xi(0.5 \pm I * t)\right)\right|$$

$$= \left|\sqrt{\Re\left(\xi(s' \pm I * t)\right) - \sqrt{\Re\left(\xi(0.5 \pm I * t)\right)}}\right| * \left|\sqrt{\Re\left(\xi(s' \pm I * t)\right) + \sqrt{\Re\left(\xi(0.5 \pm I * t)\right)}}\right|$$

$$< \left|\sqrt{\varepsilon + \Re\left(\xi(s' \pm I * t)\right) - \sqrt{\Re\left(\xi(0.5 \pm I * t)\right)}}\right| * \left|\sqrt{\varepsilon + \Re\left(\xi(s' \pm I * t)\right) + \sqrt{\Re\left(\xi(0.5 \pm I * t)\right)}}\right|$$

$$< \left|\varepsilon + \Re\left(\xi(s' \pm I * t)\right) - \Re\left(\xi(0.5 \pm I * t)\right)\right|$$

= ε.)

(N.B. If we may draw an arbitrary line first and then move the real and imaginary curves until their intersection points meet the line, we may get a similar result just like the case of the moving line to meet the real & imaginary curves. In particular, if we fix the line Z = 0 and move the real & imaginary curves at the same time, then the intersection points of these two curves which just meet at the fixed line Z = 0 may thus form a new set of so-called "Artificially Human Made Riemann Non-Trivial Zeta Zeros which is a very interesting outcome. Hence, if we recognize those artificially made non-trivial zeros as the normal & natural non-trivial zeta zeros, then the Riemann Hypothesis fails immediately since we find that the intersection points between real and imaginary curves for the real part $x \neq 0.5$ or in the other words – we still have some artificially human made non-trivial zeta zeros outside the critical line x = 0.5 by just shifting both of the real and imaginary curves. Otherwise, the Riemann Hypothesis still hold.)

(N.B. By the method of mirror imaged "Inverse (I.e. vice versa) Optimization", if we were given a set of z = (x,y) value, we might find its mirror imaged (or the inverse mapping/mapped) complex number in the critical strip region (where x = 1 is the pole being excluded) for a further mathematical study to determine if there were any other feasible non-trivial roots, i.e. rather than the critical strip x = 0.5, of the Riemann Zeta function.) To be precise, we may find the present (mirror imaged inverse optimization's) initial (optimization) conditions and check with the world recognized Riemann Zeta function's optimization condition for a comparison. Then we may see if there may be a need for any further adjustment in those of the well-known Riemann Zeta Optimization conditions etc.

A Boundary between convergent & divergent

In practice, rather than the scaled proof for the Riemann Hypothesis, there is actually a boundary between the convergent and divergent for the Riemann Zeta function. With reference to the U.S.A. commercial software – Mathematica, we may generalize the situation into three categories as listed in the next page according to their parametric plot of the Newton Flow for the Riemann Xi-function. In reality, the program code segments contain the symbolic definition of the Xi-function together with the differentiation of the Xi-function etc. When we are discussing about the parametric plot, we have used the previously defined newton Flow function and employing the newton Flow through the rainbow color distribution for a detailed research and investigation.

Case I: Zeta (x + y*I) where 0 < x < 1 and is divergent

```
In[•]:= xi[z_] := Zeta[1/2 + I*z] * (Zeta[1/2 - I*z]); (*This is an example,
        use the correct xi function from a library if available*)
        f[z_] = xi[z];
        df[z_] = D[f[z], z];
        newtonFlow[z_] = -f[z] / df[z];
        ParametricPlot[{Re[newtonFlow[x + I*y]], Im[newtonFlow[x + I*y]]}, {x, -5, 5},
         {y, -5, 5}, (*Adjust domain as needed*) PlotRange → All, ColorFunction → "Rainbow",
         PlotLabel → "Newton Flow of Riemann xi-Function"]
Out[•]=
```

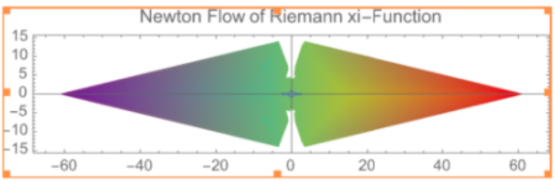


Figure 5. Newton Flow for Riemann xi function at the critical strip region 0 < x < 1

From the above figure about the parametric plot at the Riemann zeta critical strip region 0 < x < 1, one may observe that the Newton Flow of the Riemann xi-Function will converge at both ends of the right and left. In fact the Newton Flow will achieve its highest point when it is approaching or sandwiching to zero from both sides with a sudden drop at zero (which is supposed to be those non-trivial zeta zeros). As the rainbow color at both of the converging ends seem to be in deeper one (deep blue or deep red) than the rest parts, this author proposes that the flow density will become extremely condensed while as the higher as the other end near zeros, the Newton Flow density seems to be less denser as the rainbow color are likely to be lighter (green or yellow). In reality, the Newton Flow of a complex function F(z) looks like the Riemann Zeta, one can always describe the Newton Flow by the differential equation:

$$\frac{dz}{dt} = -\frac{F(z)}{F'(z)}$$

where z is a complex variable and t is a real parameter representing time. The Riemann xi function, ξ(z), is an entire function closely related to the Riemann Zeta function, ζ(z). In practice, The zeros of Riemann xi function, ξ(z), correspond to the non-trivial zeros of the Riemann Zeta function, ζ(z).

Case II: Zeta (x + y*I) where x = 1 (it is the last point of divergent or) which may be considered as the boundary between the divergent and the convergent

```
In[∘]:= xi[z_] := Zeta[1 + I * z] * (Zeta[1 - I * z]); (*This is an example,
        use the correct xi function from a library if available*)
        f[z_] = xi[z];
        df[z_] = D[f[z], z];
        newtonFlow[z_] = -f[z] / df[z];
        ParametricPlot[{Re[newtonFlow[x + I * y]], Im[newtonFlow[x + I * y]]}, {x, -5, 5},
         {y, -5, 5}, (*Adjust domain as needed*) PlotRange → All, ColorFunction → "Rainbow",
         PlotLabel → "Newton Flow of Riemann xi-Function"]
```

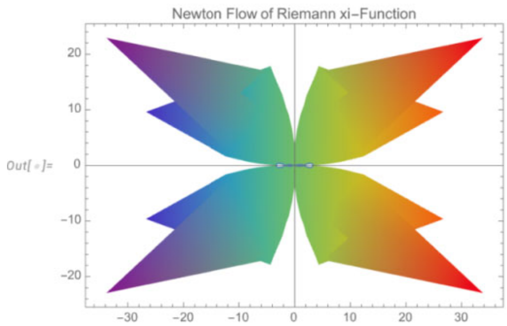


Figure 6. Newton Flow for Riemann xi function at the boundary x = 1

From the above figure about the parametric plot at the Riemann Zeta function's boundary at x = 1, one may observe that the Newton Flow of the Riemann xi function appears to look like as a flying butterfly. Actually, x = 1 is a pole of the Riemann Zeta function with a sudden drop to a negative infinity (just like a cliff and may be used as an impulsed input to the digital signal processing system for checking the system responses in the output etc.) In practice, for the Riemann Zeta function at
x = 1,

the function turns to be log |x| which is just the integral of:

$$\int_a^b \frac{1}{x}\, dx + \int_c^d \frac{1}{x}\, dx = ln(-b) - ln(-a) + lnd - lnc$$

where a = 1^- and b = 0^- & c = 0^+ and d = 1^+ with $ln(0) = -\infty$

$$= ln(b) - ln(a) + lnd - lnc$$

This author wants to note that those trajectories of the Newton Flow system's trace lines for the phase (argument) of the xi-function is a constant and may be known as the flow lines. On the contrary to the zeta function, xi function has no poles, the flow lines start at infinity and terminate at zeros of the function which acts as the "sinks" for the flow. For some special flow lines, called separatrices, these flow lines will divide the complex plane into several regions with each region associated with a specific zero of function.

Case III: Zeta (x + y*I) where x > 1 which is convergent

```
In[∘]:= xi[z_] := Zeta[2 + I * z] * (Zeta[2 - I * z]); (*This is an example,
        use the correct xi function from a library if available*)
        f[z_] = xi[z];
        df[z_] = D[f[z], z];
        newtonFlow[z_] = -f[z] / df[z];
        ParametricPlot[{Re[newtonFlow[x + I * y]], Im[newtonFlow[x + I * y]]}, {x, -5, 5},
         {y, -5, 5}, (*Adjust domain as needed*) PlotRange → All, ColorFunction → "Rainbow",
         PlotLabel → "Newton Flow of Riemann xi-Function"]
```

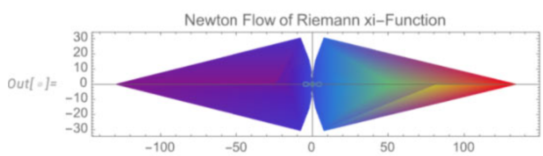


Figure 7. Newton Flow for Riemann xi function at the region x > 1

In the above figure 7, it describes the Newton Flow for the Riemann xi function at the region x > 1. Actually, the parametric plot of the flow convergent density is similar to the critical strip region except that in the middle of the right hand side triangle, the rainbow seems to be much lighter. In additional, the squeezed middle point at zero looks like approaching to a point where there is a difference between the critical strip with some thickness.

This author wants to note that the Newton Flow provides a geometric representation of the complex plane and allowing for the visualization of the relationships between the function's zeros and its phase. In reality, the Riemann Hypothesis conjectures that all non-trivial zeros of the xi-function lie on the "critical line" with the real part equals to 0.5. Lastly, the behavior of the Newton flow, particularly the structure of its separatrices and where they lead can help us to study the way of finding insights into this problem.

(N.B. The above three conditions for the convergence / divergence are just the counter-example to the normal case of sequence / series' radius of convergence such that: Case I: 0 < R <1 implies converge; Case II: R = 1 is un-determined; Case III: R > 1 implies diverge.)

Conclusions

In the region where 0 < Re(z) < 1 which is just well known as the critical strip region for the Riemann Hypothesis. It is also supposed that all of the non-trivial zeros only lie on the x = 0.5. However, with reference to my previous research outcome in the former section, all of the other x = x' (such as 0.1 or 0.7) in the critical strip region may also have their own intersection points for Re(z') and Im(z') which are just the angular rotation to the intersection points for x = 0.5 (i.e. Re(z) and Im(z)) correspondingly. In other words, there may be a controversy about whether all of the non-trivial zeros must lie on the line x = 0.5. Or one may find that, after a suitable coordinate transformation or a phase shifting of an angle theta, these abnormal non-trivial zeta zeros (intersection points between Re(z') and Im(z')) will overlap with those normal non trivial zeta zeros (intersection points between $Re(z_{0.5}) = Im(z_{0.5})$). Hence, one may consider these abnormal non-trivial zeta zeros to be the same as the those normal non-trivial zeta zeros. Then the Riemann Hypothesis is said to be correct. On the other hand, from the numerical value's point of view, these abnormal non trivial zeta zeros are actually different from the normal non-trivial zeta zeros. In my opinion, there are actually infinite many non-trivial zeta zeros (both the normal and abnormal ones) lying over the critical strip region 0 < x < 1 but not just the line x = 0.5, that is in practice, without any coordinate transformation or angular rotation for those concerned complex numbers' theta. Then the Riemann Hypothesis is said to be incorrect in such case (if we have just relaxed the requirement x = 0.5 is in an optimal status). However, as shown in my previous proof, only those normal non-trivial zeta zeros that lies on the line x = 0.5 are found to be in an optimal status. Therefore in such a case, this author will definitely conclude that Riemann Hypothesis must be true for those non-trivial zeta zeros lie on the optimal line x = 0.5 in the critical strip region 0 < x <1 while those other abnormal non-trivial zeta zeros (such as the lines x = Re(z') = 0.1 or Re(z') = 0.7 etc) may not be the most optimal one as the conditions historically required in the Riemann Hypothesis statement which is supposed there will certainly be a proof every time.

In addition, if we apply the Hessian Matrix to the Lagrange Multiplier of the Re(ξ(x+It)) equation's expansion like the following:

$$-80\left(u^4+12*(n/3+1)\right)*u^3+51*((n+1)^2-20*(n+1)*n/17+(7*n^2)/17)*u^2$$
$$+(90*(n+1)^4-218*(n+1)^3*n+210*(n+1)^2*n^2-102*(n+1)*n^3+20*n^4)*u$$
$$+(24*n^4)/5+(274*(n+1)^4)/5-(326*(n+1)^3*n)/5+(274*(n+1)^2*n^2)/5-126*(n+1)*n^3/5$$

subject to the constraint "u*n" and $\lambda = -9*10^{-6}\sqrt{2}$, then according the Maple Soft code below:

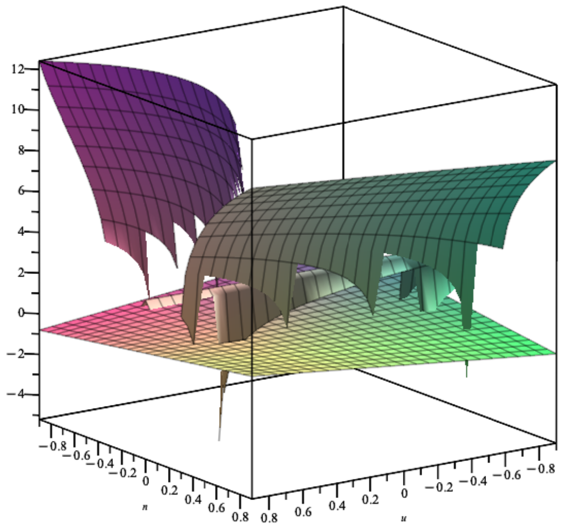
```
with(VectorCalculus);

L := arg3 - lambda*(n*u – 1);
H, detH := Hessian(L, [lambda, u, n], determinant);
eval(detH, [u = 1, n = 1, lambda = -9.*10^(-6)*sqrt(2)]);
```

The final output value is -424244.2749 < 0. It is obviously implying that x = 0.5 is just what we so-called the "Saddle-point". Or in the sense of quantum physics harmonic oscillating as described in my previous paragraph, x = 0.5 is the optimal point (in reality, the saddle point) of all harmonic equilibrium points among the critical region 0 < x <1, i.e. all of the harmonic equilibrium points between 0 < x <1 has only one saddle point at x = 0.5.

In brief, no matter we consider the normal non trivial zeta zeros to be equal to the abnormal non-trivial zeta zeros or not, Riemann Hypothesis will still be proved by this author to be true unless we relax the optimal condition requirement in the Riemann Hypothesis statement. Thus, this author have just successfully ended the controversy about the truth-less of the Riemann Hypothesis.

Appendix: 3D view of the Step like Structure for -0.9 < x < 0.9 & -0.9 < y < 0.9 by Maple Soft code:

```
plot3d([ln(arg3), u*n], u = -0.9 .. 0.9, n = -0.9 .. 0.9)
```

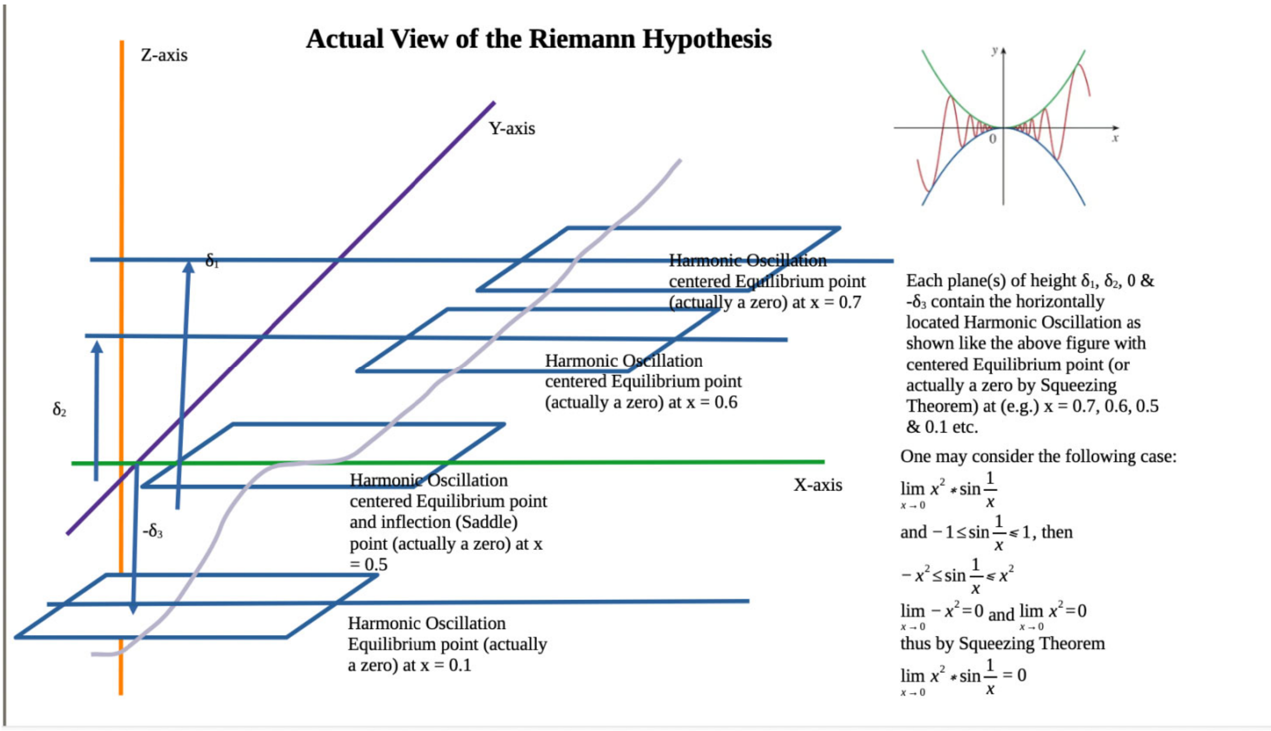


(N.B. In fact, for the Hessian matrix value = -424244.2749, if one takes logarithm and changes it into the complex number, one may get the value equals to: 12.95806469+$\pi * I$. For the second eigenvalue $\pi * I$, this implies that it is an unstable spiral or a saddle-focus (in 3D or complex space). The real part, 12.95806469, Indicates an Unstable direction. Trajectories will be pushed away from the point along this axis.)
(N.B. In practice, there is a (mirror) inverse Hessian matrix which may be referred to the formation of the corresponding Fourier series and hence to go a step for the respective filter equation in the field of the digital signal processing. However, the focus of the present paper is in the zoomed proof of the Riemann Hypothesis and the critical line x = 0.5 which is in fact the most optimized one (or the saddle point) of all equilibrium points (or the vice versa way) for the harmonic oscillation in the sense of quantum mechanics. Also, the Fourier Hessian Matrix may be primarily used for the linearized equation as an approximation. To go a step, one may apply the Fourier Hessian Matrix as a way for the deconvolution of an image in the digital signal processing together with the inverse Fourier Hessian Matrix as a method for the convolution of an image in the DSP etc.)

(N.B. We may improve the efficiency (or the research gap) of finding the saddle point from the Hessian matrix by gradient descent through both of the optimization together with the iterative progress like the following (Maple Soft code):

```
restart;

with(VectorCalculus):
with(Optimization):
with(LinearAlgebra):

# 1. Define the objective function
f := (x, y) -> 10*(y-x^2)^2 + (1-x)^2; # Rosenbrock Function

# 2. Iterative optimization parameters
initial_guess := Vector([1.2, 1.2]);
tol := 1e-5;
max_iter := 20;

# 3. Initial Hessian Approximation (Identity Matrix)
B := IdentityMatrix(2);
x_curr := initial_guess;

# 4. Iterative Loop
for i from 1 to max_iter do

# Calculate Gradient at current x
grad := Vector([diff(f(x, y), x), diff(f(x, y), y)]);
g_curr := subs({x=x_curr[1], y=x_curr[2]}, grad);

# Check convergence (if norm of gradient is small)
if Norm(g_curr, 2) < tol then
print(Converged_at_iteration = i);
break;
end if;
```

```
# Calculate search direction (Newton step: -H^-1 * g)
p := -LinearSolve(B, g_curr);

# Update position (simple line search for example)
x_next := x_curr + 0.1 * p;
# Update Hessian Approximation (BFGS Update Formula)
s := x_next - x_curr;
g_next := subs({x=x_next[1], y=x_next[2]}, grad);
y_diff := g_next - g_curr;

# BFGS update formula
rho := 1 / DotProduct(y_diff, s);
B := B + rho * OuterProduct(y_diff, y_diff) - (1/DotProduct(s, B.s)) * OuterProduct(B.s, B.s);

x_curr := x_next;
print(Iteration = i, Position = x_curr);
end do;)
```



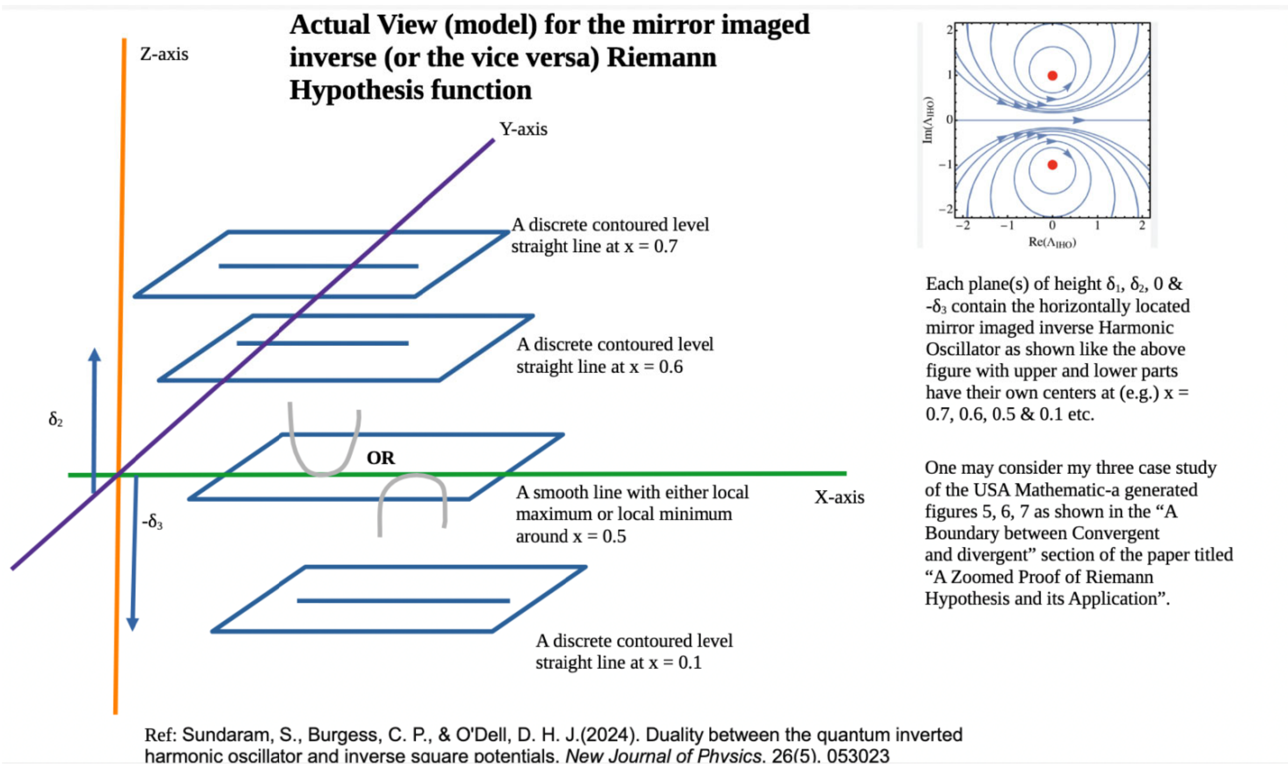
(N.B. There is a pole at x = 1 which is NOT shown in the above diagram. The grey line is NOT $y=x^3$ as it does NOT pass through origin (0,0). The focus of the grey line is it has a saddle point or the point of inflection at x = 0.5 – the critical line of the Riemann Zeta function which is just the flat area among all of the equilibrium points in the harmonic oscillation quantum mechanics for all points between 0 and 0.9.)

(N.B. The reciprocal of the equilibrium point of a harmonic oscillating (actually equal to zero) is in fact an infinity or just an impulse response which may constitute a linear time-invariant system(s).)

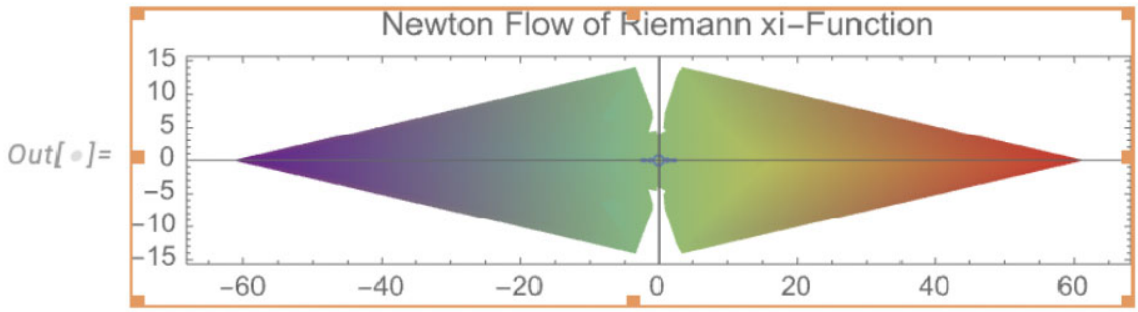
(N.B. Reciprocal of (Harmonic Oscillation of the quantum mechanics with) central equilibrium point (equal to zero) is just the intersection point between the black hole and the white hole – the so-called "wormhole" which may be described by a linear time invariant system. By the way, there may be an inverse linear time invariant system which is just the inverse of the linear time invariant system for the harmonic oscillation assumption(s) at the early beginning of the above actual view.)

Ultimately, this author wants to note that from the "Actual View of the Riemann Hypothesis" diagram, one may observe that there is a jump from one level to another level (i.e. at the Saddle point), then according to the Morera's Theorem in the complex analysis, the function f (i.e. the grey line) connecting the real part of the Zeta function must be holomorphic between [0, 0.9]. This property implies the function f must be analytic and continuous. Hence, f must have a high regularity and by the Cauchy's Integral Formula, f must be infinitely differentiable and equal to its Taylor series locally. Or this author's previous work(s) in Riemann Hypothesis through the Taylor Series Expansion applied to the Riemann Zeta Function should be logical and acceptable.

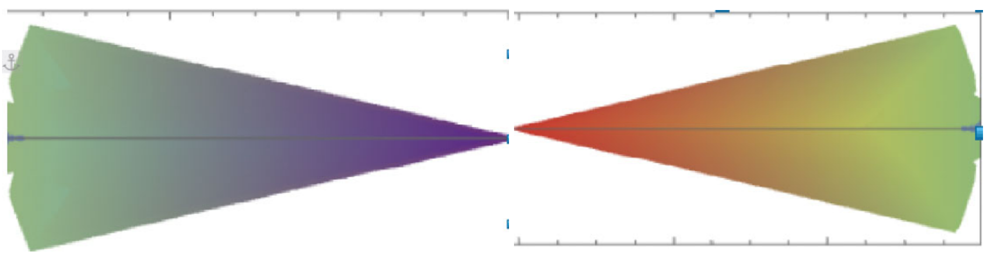
Appendix: Refer to my Actual View (Model) for the Riemann Hypothesis



(N.B.By the way, if the left hand cone and the right hand cone are inverted horizontally, i.e. From:



Into:



The above figure has just shown how the black hole meets with the white hole such that the intersection point in the middle as the worm hole.)

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