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Research Article

FUZZY DOT *BCK*- FILTERS

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Abstract

The concept of fuzzy dot subalgebras of *BCK/BCI*-algebras was introduced by Jun and Hong (Jun *et al.,* 2001). In this paper, we define the notion of fuzzy dot filters of *BCK*-algebras and prove some theorems.

Keywords: *BCK*-algebras, filter, fuzzy filter, fuzzy dot filter.

INTRODUCTION

The notion of *BCK*-algebra was introduced by Imai and Iseki in 1966. In same year, Iseki (1966) introduced the notion of *BCI*-algebra which is a generalization of a *BCK*-algebra. After the introduction of the concept of fuzzy subsets by Zadeh (Xi, 1991), several researchers worked on the generalization of the notion fuzzy sets. Jun and Hong (2001) introduced the notion of a fuzzy dot subalgebra of a *BCK/BCI*-algebra as a generalization of a fuzzy subalgebra and prove several basic properties which are related to fuzzy dot subalgebra. In this paper, we introduce the notion of fuzzy dot filter and prove some their fundamental properties.

2. Preliminaries

We review some basic definitions and properties that will be useful in our results. A *BCK-algebra* X is defined to be an algebra $(X,*,0)$ of type $(2,0)$ satisfying the following conditions (Ise'ki, 1966; Iseki and Tanaka, 1976):

$BCK-1$ $((xy)(xz))(zy)=0,$

$BCK-2$ $(x(xy))y=0,$

$BCK-3$ $xx=0,$

$BCK-4$ $0x=0,$

$BCK-5$ $xy=0$ and $yx=0 \Rightarrow x=y,$

for all $x,y,z \in X$, where $xy=x*y$, and $xy=0$ if and only if $x \le y$.

In a *BCK*-algebra X, the following properties hold for all $x,y,z \in X$:

P-1 $x0=x$.
P-2 $(xy)z=(xz)y$.
P-3 $x \le y$ implies that $xz \le yz$ and $zy \le zx$.
P-4 $(xz)(yz) \le xy$.
P-5 $x \le y$, $y \le z \Rightarrow x \le z$.

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A *BCK*-algebra X satisfying the identity $x(xy)=y(yx)$, for all $x,y \in X$, is said to be *commutative* (Deeba, 1979). If there is a special element 1 in a *BCK*-algebra X satisfying $x \le 1$, for all $x \in X$, then 1 is called a *unit* of X (Deeba, 1979). A *BCK*-algebra X with unit is said to be *bounded.* In a bounded *BCK*-algebra X, We have (Iseki and Tanaka, 1976):

P-6 $1^*=0$ and $0^*=1$.
P-7 $y \le x$ implies that $x^* \le y^*$.
P-8 $x^*y^* \le yx$.
where $1x=x^*$.

If X is a commutative bounded *BCK*-algebra, then the identity $x^*y^*=yx$ holds, for all $x,y \in X$, (Iseki and Tanaka, 1976).

We review some fuzzy concepts. A fuzzy subset of a nonempty set X is a function $\mu : X \to [0,1]$. The set $\mu_t = \{x \in X \mid \mu(x) \ge t\}$, where $t \in [0,1]$, is called the *t-level subset* of μ [14].

In 1979, E. Deeba (1979) defined a filter (also called dual ideal) in a *BCK*-algebra as:

A nonempty subset F of a *BCK*-algebra X is called a *filter* if:

(i) $x \in F$, $x \le y$ imply $y \in F$,
(ii) $x \in F$, $y \in F$ imply there exists an element $z \in F$ such that $z \le x$, $z \le y$.

In (Ahmed, 1982), filters have been characterized in a commutative *BCK*-algebras. For the basic properties of filters, we refer to (Ahmad, 1982; Ahmad, 1982; Deeba, 1979; Meng, 1996).

In (Liu and Meng, 2002), J. Meng defined filter in a bounded *BCK*-algebras as:

A nonempty subset F of a bounded *BCK*-algebra X is called a *filter* of X if it satisfies:

(F-1) $1 \in F$,
(F-2) $(x^* y^*)^* \in F$ and $y \in F$ imply $x \in F$, for all $x, y \in X$.

A fuzzy subset μ of a bounded *BCK*-algebra X is said to be a *fuzzy filter* of X, if it satisfies:

(FF-1) $\mu(1) \geq \mu(x)$,
(FF-2) $\mu(x) \geq \min\{\mu((x^* y^*)^*), \mu(y)\}$, for all $x, y \in X$ (Jun and Hong, 1988).

FUZZY DOT FILTERS

Throughout this paper X is a bounded *BCK*-algebra, unless otherwise stated. First, we define:

Definition 3.1. Let μ be a fuzzy subset of X. Then μ is called a *fuzzy dot filter* of X, if it satisfies (FF-1) and the condition.
(FF-3) $\mu(x) \geq \mu((x^* y^*)^*)\mu(y)$, for all $x, y \in X$.

Example 3.2. Let $X = \{0, a, b, 1\}$ be a bounded *BCK*-algebra with $*$ defined by

*	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

Define μ of X by $\mu(1) = 0.7$ and $\mu(0) = \mu(a) = \mu(b) = 0.5$. Routine calculations give that μ is a fuzzy dot filter of X. Also, a fuzzy subset ν of X, defined by $\nu(0) = 0.3$, $\nu(a) = 0.4$ and $\nu(b) = \nu(1) = 0.5$. Calculations give that ν is a fuzzy dot filter of X.

Remark 3.3. Every fuzzy filter of X is a fuzzy dot filter of X, since $\mu(x) \geq min\{\mu((x^* y^*)^*), \mu(y)\} \geq \mu((x^* y^*)^*)\mu(y)$, but the converse may not be true as is seen in the above example, since ν is a fuzzy dot filter of X, but it is not a fuzzy filter

$$\nu(a) = 0.3 < min\{\nu(0^* a^*)^*, \nu(a)\} = min\{\nu(b), \nu(a)\} = 0.4$$

Proposition 3.4. *Every fuzzy dot filter* μ *of* X with $\mu(1) = 1$, is order preserving.

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $y^* \leq x^*$ and so $y^* x^* = 0$. Hence

$$\mu(y) \geq \mu(y^* x^*)^* \mu(x)$$
$$= \mu(1)\mu(x)$$
$$= \mu(x).$$

Proposition 3.5. Let μ be a fuzzy dot filter of a *BCK-algebra* X, and $\mu(1) = 1$. Then for all $x, y, z \in X$, it satisfies the condition (1) $\mu(xy) \leq \mu((xy)y)$, if and only if it satisfies (2) $\mu((xz)(yz)) \leq \mu((xy)z)$.

Proof. Let μ be a fuzzy dot filter of X satisfying (1). Since $((x(yz))z)z = ((xz)(yz))z \leq (xy)z$, by Proposition 3.4, we have $\mu(((x(yz))z)z) \leq \mu((xy)z)$. It follows from (1) that.

$$\mu((xz)(yz)) = \mu((x(yz))z)$$
$$\leq \mu(((x(yz))z)z)$$
$$\leq \mu((xy)z).$$

Thus μ satisfies (2).
Conversely, replacing z with y in (2), we obtain the condition (1). This completes the proof.

Proposition 3.6. Let μ be a fuzzy dot filter of X. Then for all $x \in X$:

(i) $\mu(0) \geq \mu(x^*)\mu(x)$,
(ii) $\mu(x) \geq \mu(1)\mu(0)$,
(iii) If $\mu(1) = 1$, then $\mu(xy) \geq \mu(x)\mu(y^*)$.

Proof. Let $x \in X$. (i) By definition μ, we have

$$\mu(0) \geq \mu(0^* x^*)^* \mu(x)$$
$$= \mu(1x^*)^* \mu(x)$$
$$= \mu(x^{***})\mu(x)$$
$$= \mu(x^*)\mu(x)$$

(ii) $\mu(x) \geq \mu(x^* 0^*)^* \mu(0)$
$$= \mu(x^* 1)^* \mu(0)$$
$$= \mu(0^*)\mu(0)$$
$$= \mu(1)\mu(0)$$

(iii) Since $((xy)^* x^*) \leq x(xy) \leq y$, then $y^* \leq ((xy)^* x^*)^*$, so by Proposition 3.4 $\mu(y^*) \leq \mu((xy)^* x^*)^*$. Thus

$$\mu(xy) \geq \mu((xy)^* x^*)\mu(x)$$
$$\geq \mu(y^*)\mu(x) = \mu(x)\mu(y^*).$$

Theorem 3.7. *Let* μ *be* a fuzzy subset of X and $\mu(1) = 1$. Then
(i) $\mu(xz)^* \geq \mu(y)$ implies $\mu(z) \geq \mu(y)$.
(ii) $xy \leq z$ implies $\mu(y) \geq \mu(x)\mu(y^*)$.

Proof. (i) Let $\mu(xz)^* \geq \mu(y)$, for all $x, y, z \in X$. Then $\mu(z^{**}) = \mu(1z)^* \geq \mu(y)$. But $z \geq z^{**}$. Then $\mu(z) \geq \mu(z^{**}) \geq \mu(y)$.

(ii) Assume that the inequality $xy \le z$ holds in X. Since $y^*x^* \le xy \le z$, and so $z^* \le (y^*x^*)^*$, it follows by Proposition 3.4. that $\mu(y^*x^*)^* \ge \mu(z^*)$. Hence $\mu(y) \ge \mu(y^*x^*)^* \mu(x) \ge \mu(z^*)\mu(x) = \mu(x)\mu(z^*)$.

Theorem 3.8. *Let X be commutative and μ a fuzzy subset of X. Then μ is a fuzzy dot filter if and only if it satisfies for all $x, y \in X$,*

$$\mu(x) \ge \mu(yx)^* \mu(y) \qquad (3)$$

Proof. Since $x^*y^* = yx$, and so $(x^*y^*)^* = (yx)^*$. Then μ is a fuzzy dot filter of X $\Leftrightarrow \mu(x) \ge \mu(x^*y^*)^* \mu(y) \Leftrightarrow \mu(x) \ge \mu(yx)^* \mu(y)$.

Theorem 3.9. *Let F be a filter of X. Let μ_F be a fuzzy subset of X, defined by $\mu_F(x) = s$ if $x \in F$ and $\mu_F(x) = t$ if $x \notin F$, for all $s, t \in [0,1]$ with $s > t$. Then μ_F is a fuzzy dot filter of X.*

Proof. Let F be a filter of X. Since $1 \in F$, we have $\mu(1) = s \ge \mu_F(x)$, for all $x \in X$. Now let $x, y \in X$, if $x \in F$, then $\mu_F(x) = s \ge \mu_F(x^*y^*)^* \mu_F(y)$. If $x \notin F$, then $(x^*y^*)^* \notin F$ or $y \notin F$, then $\mu_F(x) = t \ge \mu_F(x^*y^*)^* \mu_F(y)$, it follows that μ_F is a fuzzy dot filter.

Proposition 3.10. *Let μ be a fuzzy dot filter of X. Then $X_\mu = \{x \in X \mid \mu(x) = 1\}$ is a filter of X.*

Proof. Suppose that μ is a fuzzy dot filter of X. Clearly $1 \in X_\mu$, now let $(x^*y^*)^* \in X_\mu$ and $y \in X_\mu$, then $\mu(x^*y^*)^* = 1 = \mu(y)$, so $\mu(x) \ge \mu(x^*y^*)^* \mu(y) = 1$, so $x \in X_\mu$, then X_μ is a filter of X.

Theorem 3.11. *Let $\{\mu_i\}$, where $i \in I$ be a family of fuzzy dot filters of X. Then so is $\bigcap_{i \in I} \mu_i$.*

Proof. For all $x, y \in X$, we get.

$$\bigcap_{i\in I} \mu_i(1) = \min_{i\in I}\{\mu_i(1)\}$$
$$\ge \min_{i\in I}\{\mu_i(x)\}$$
$$= \bigcap_{i\in I} \mu_i(x)$$
$$\bigcap_{i\in I} \mu_i(x) = \min_{i\in I}\{\mu_i(x)\}$$
$$\ge \min_{i\in I}\{\mu_i(x^*y^*)^* \mu_i(y)\}$$
$$\ge \left(\min_{i\in I}\{\mu_i(x^*y^*)^*\}\right)\left(\min_{i\in I}\{\mu_i(y)\}\right)$$
$$= \left(\bigcap_{i\in I} \mu_i(x^*y^*)^*\right)\left(\bigcap_{i\in I} \mu_i(y)\right)$$

Hence $\bigcap_{i\in I} \mu_i$ is a fuzzy dot filter of X.

In the following example, we can see if $\{\mu_i\}$, where $i \in I$ is a family of fuzzy dot filters of *a BCK-algebra X, then* $\bigcup_{i\in I} \mu_i$ may be not a fuzzy dot filter of X.

Example 3.12. Suppose $X = \{0, a, b, 1\}$ is a bounded *BCK*-algebra given in Example 3.2. Let a fuzzy dot filter ν of X be defined by $\nu(0) = 0.3$, $\nu(a) = 0.4$ and $\nu(b) = \nu(1) = 0.5$. Define the fuzzy subset λ of X by $\lambda(1) = \lambda(a) = 1$ and $\lambda(0) = \lambda(b) = 0.1$. Routine calculations give that λ is a fuzzy dot filter of X. *But* $\nu \cup \lambda$ is not a fuzzy dot filter, because.

$$\lambda \cup \nu(0) = \max\{\lambda(0), \nu(0)\} = 0.3$$
$$< \lambda \cup \nu(0^*b^*)^* \lambda \cup \nu(b)$$

$$= (max\{\lambda(0^*b^*)^*, \nu(0^*b^*)\})(max\{\lambda(b), \nu(b)\})$$
$$= (max\{1, 0.4\})(max\{0.1, 0.5\}) = (1)(0.5) = 0.5.$$

Remark 3.13. Note that a fuzzy subset μ of X is a fuzzy filter if and only if nonempty level subset μ_t is a filter for all $t \in [0,1]$. But a fuzzy dot filter may not be a fuzzy filter (see Example 3.2.). If μ is a fuzzy dot filter of X, then μ_t may not be a filter. In Example 3.2., if we take $\nu_{0.4} = \{a, b, 1\}$, then $(0^*a^*)^* = (1b)^* = b \in \nu_{0.4}$ and $a \in \nu_{0.4}$ but $0 \notin \nu_{0.4}$, so $\nu_{0.4}$ is not a filter, while ν is a fuzzy dot filter of X.

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